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Generalized plane strain analysis of superconducting solenoids

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A stress analysis of superconducting solenoids is presented which includes a generalized plane strain (GPS) condition for the axial strain. The GPS condition is introduced on the assumption that the deformation of a solenoid from a right circular cylinder is small. The GPS assumption results in an analytic solution for all three components of stress and strain in a solenoid. The work is presented in the context of the historical development of stress analysis for solenoids. The general stress equations for a magnetic solenoid are formulated. The relationship between a right cylinder deformation and the generalized plane strain condition is examined for the physical conditions in the central region of a solenoid magnet. The general analytic solutions of the stress equations are given for the cases of magnetic and thermal loading. The constant coefficients are determined for cases of common interest in solenoid magnet design. The analytic results are compared with numerical analysis results for an example solenoid consisting of a single coil with external reinforcement. In particular, the degree to which the axial strain is a constant and satisfies the GPS assumption is examined for the example solenoid. The analysis reveals features of the axial stress in solenoids, including the Poisson's ratio induced axial stress and the axial stress distribution between coil and reinforcement during cooldown and operation. The strong agreement between the GPS and numerical analysis results shows that the assumptions contained in the GPS analysis accurately represent the conditions in the central region of a solenoid magnet. [S0021-8979(99)01024-5]

I. INTRODUCTION

The mechanical stress analysis of superconducting solenoid magnets is an essential and integral part of the design process. The windings of a high field solenoid are typically a complex composite material of conductor, reinforcement, and insulation. As magnets increase in field and bore size, the windings are subjected to increasing values of mechanical stress, and a more complete understanding of the stress distribution is required. Here, the detailed formulation and solution of a three dimensional analysis of stress in solenoid magnets under the limiting assumption of generalized plane strain (GPS) is presented.

A solenoid magnet is a cylindrical structure with a nonuniform distributed body force. A magnet may be constructed from a number of coils of increasing diameter nested together and, as will be assumed here, mechanically independent except for alignment. A magnet can be a single coil as well. Within a coil, the dominant magnetic force is radial outward. In addition, there is a significant load from the axial component of magnetic force which is distributed primarily near the ends of a coil. The problem of stress analysis in solenoid coils is made difficult by a degree of bending, especially toward the ends of a coil, which has an associated shear stress, and by the lack of a closed analytic form for the distributed magnetic load, which originates from the field produced by the entire set of coils which constitute the magnet. Historically, the analytical treatments have been restricted to two dimensions with assumptions suitable only to the coil midplane. The dominant radial magnetic force gives rise to a reaction stress in the tangential, or hoop, direction. The two dimensional analyses treat the tangential and radial components of stress. But the actual stress distribution, even in the midplane, is inherently three dimensional. As an ideal limiting condition, it is assumed that the central region about the midplane of a solenoid coil is in a state of generalized axial plane strain. It is found that with the GPS assumption, the resulting stress equations can be solved directly to yield a solution which displays the essential aspects of the full three dimensional conditions at the solenoid midplane.

A superconducting magnet must be cooled to low temperature for operation. An anisotropy in the thermal contraction of the windings, or a difference in thermal contraction between windings and reinforcement will result in a mechanical stress. That mechanical stress which results from thermal contraction is called the thermal stress. The GPS assumption is applied to the thermal stress, and as with the mechanical stress from magnetic loads, a three dimensional solution is obtained. Within a linear elastic model, the total stress in a solenoid is the superposition of the mechanical and thermal stress.

Here, in a systematic way, the form of the most general stress balance equations for a cylindrical solenoid magnet are first formulated. The assumption of GPS is then shown to be

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related to the displacement and loading in the central region of a solenoid. Under the assumption of GPS, the stress balance equations for the mechanical stress problem and for the thermal stress problem are formulated. The general solutions to these equations are first expressed in terms of constant coefficients. To complete the full analytical solutions, the detailed derivations of the constant coefficients for configurations of usual interest are then given.

II. MATERIALS PROPERTIES

This stress analysis applies to the windings of a cylindrical solenoid. It is assumed that the principle material directions are aligned with the cylindrical coordinate axes. Coils may be layer wound of round or rectangular conductor. In a helical winding, the cylindrical coordinate axes are an approximation to the principle material axes in any layer. For many coil constructions, the pitch angle of the windings along the cylindrical axis is zero except for a short arc length in which a jog occurs to move the wire along the axis. Except in these jog regions, the wire is aligned with the coordinate axes. Coils may also be pancake wound of tape conductor, in which case the conductor is always aligned with the coordinate axes except for localized connections at the inner and outer radius of the coil.

The materials properties in the direction along the conductor are dominated by the properties of the conductor itself. The analysis assumes a uniform current density, which is the case for a coil wound of a uniform cross section conductor with a uniform distribution of insulation. The insulation in a coil tends to have a lower elastic modulus than the conductor. As a result, the materials properties in the directions transverse to the conductor can be strongly influenced by the insulation. Furthermore, it is not uncommon to have a different insulation system between layers than between turns, in the form of an insulation sheet or cloth between layers in addition to the insulation on the individual wires. These factors are reflected in the assumption that the windings are a homogeneous and orthotropic material in the principal material axes, for both mechanical properties and thermal contraction.

The windings of a coil may be reinforced. The reinforcement can take the form of a shell around the cylindrical windings or may be distributed among the windings. An external shell may be a homogeneous solid cylinder or may be a winding of reinforcement wire similar to the conductor windings. The reinforcement is likewise assumed to be homogeneous and orthotropic.

The windings of a coil are thus a composite material consisting of conductor, insulation, and possibly distributed reinforcement. The principles of macromechanics of composites are well suited to the computation of the average material properties of the windings.[1] The stress analysis is formulated in terms of the average materials properties, and the results are the average stress and strain.

In the principle material coordinates, the general form of the stress–strain relations for an orthotropic material is

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{31}\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{21} & C_{31} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{32} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{bmatrix} \tag{1}$$

in a commonly used mixture of contracted notation for the normal stress and strain (σ,ϵ) and engineering notation for the shear stress and strain (τ,γ), and where the C_{ij} are the components of the stiffness matrix. In the formulation of the solenoid stress problem, it is seen that there is no coupling between the normal stress and the shear strain. Each component of the shear strain is related to the component of shear stress in the same plane.

The compliance matrix formulation is most useful to obtain the compliance matrix elements from the engineering constants. The symmetric compliance matrix for an orthotropic material is given by

$$S_{ij}=\begin{bmatrix}\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0\\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0\\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}}\end{bmatrix}, \tag{2}$$

where the E_i are Young's moduli, the G_{ij} are shear moduli, and the $\nu_{ij}=-\epsilon_j/\epsilon_i$, are Poisson's ratios. The stiffness $\mathbf{C}$ is obtained from the compliance $\mathbf{S}$ by matrix inversion.

A thermal strain is associated with the cooldown of the superconducting coil to operating temperature. The thermal contraction strain

$$\epsilon_i^{th}=\alpha_i \Delta T \tag{3}$$

is assumed to have no shear components in the principle material axes.

III. STRESS BALANCE EQUATIONS FOR MAGNETIC SOLENOID, GENERAL FORM

In the cylindrical coordinates (r,θ,z), the uniform current density J_θ gives rise to field components B_z and B_r, which are independent of θ. The distributed Lorentz force density per unit volume of the windings has components

$$X_r=J_\theta B_z,\qquad X_z=-J_\theta B_r. \tag{4}$$

The general displacement of a point in a cylinder may be described by components (u,v,w) along the coordinate axes as a function of position throughout the cylinder. On the basis of the symmetry of a solenoid winding, the symmetry of the Lorentz force density, and the assumption of homogeneous material properties including the thermal contraction, the general form of the displacement in a magnetic solenoid under both the thermal contraction loading, and the Lorentz force loading will be

$$\begin{aligned} u &= u(r,z), \\ v &= 0, \\ w &= w(r,z). \end{aligned} \tag{5}$$

The strain–displacement relations of continuum mechanics are well known, relating the displacement to the total strain. The total strain at any point is the sum of the total mechanical strain and the thermal contraction strain. For the above form of displacement, the nonzero components of strain are

$$\begin{aligned} \epsilon_r^{\text{tot}} &= \epsilon_r + \alpha_r \Delta T = \frac{\partial u}{\partial r}, \\ \epsilon_\theta^{\text{tot}} &= \epsilon_\theta + \alpha_\theta \Delta T = \frac{u}{r}, \\ \epsilon_z^{\text{tot}} &= \epsilon_z + \alpha_z \Delta T = \frac{\partial w}{\partial z}, \\ \gamma_{rz}^{\text{tot}} &= \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \end{aligned} \tag{6}$$

where it should be noted that the total mechanical strain is the sum of the mechanical strain resulting from the magnetic loads and the mechanical strain resulting from the thermal contraction loads. In particular, it is seen that the components $\gamma_{r\theta}$ and $\gamma_{\theta z}$ of the shear strain are zero. From the form of the stress–strain relations, it is evident that in an orthotropic magnetic solenoid the shear stress components $\tau_{r\theta}$ and $\tau_{\theta z}$ are also zero.

The general stress balance equations for a body with a distributed force are well known. For a magnetic solenoid, given the above discussion, the nonzero components of stress reduce to σ_r, σ_θ, σ_z, and τ_{rz}. Accordingly, the most general form of the stress equilibrium equations for a magnetic solenoid are

$$\begin{aligned} &\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + X_r = 0, \\ &\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + X_z = 0, \end{aligned} \tag{7}$$

where, in the case of thermal stress only, the distributed mechanical load **X** is zero.

IV. HISTORICAL REVIEW

The problem of solenoid stress and strain as formulated in the above equations has been addressed over an extended period of time with increasing generality. A good survey of especially the earlier literature is given by Bobrov.[2] The major developments in the evolution of an analytical treatment are reviewed here, relating assumptions and simplifications made by various authors in the treatment of the forgoing equations.

The present work is in the line of development that is characterized by the work of Lontai and Marston.[3] In this work, the windings were assumed to be a homogeneous, linear elastic material, with constant current density subject to the distributed Lorentz body force. The equations are formulated in terms of the displacement of the windings and the relationship of the displacement to the strain. The solutions are obtained for coils with external reinforcement by applying the general solutions to each radial section and matching boundary conditions in order to determine coefficient values.

Limiting assumptions imposed by Lontai and Marston were isotropic material properties, a linear dependence of the axial magnetic field with radius, and zero shear. The axial force was also taken to be zero to yield a two dimensional plane stress solution. Later analyses examined cases of increasingly general assumptions. Many of these analyses were limited to the plane stress condition of zero axial stress.

The analysis of Burkhard[4] makes the same assumptions of isotropic material, two dimensional plane stress, and linear field distribution. In this treatment, there is an emphasis on coils of many radial sections and the formalism for matching the boundary conditions is well developed.

The mechanical properties of the conductor are quite different than the properties of the insulation between turns. There can be a significant difference in average properties of the windings depending on direction, especially longitudinal and transverse to the conductor. Gray and Ballou[5] introduced transverse isotropic material properties into a two-dimensional analysis which preserved the plane stress assumption. The analysis includes a detailed examination of the usual simplifying assumption that the decrease of the axial field through the windings is linear.

In a very comprehensive treatment, Arp[6] examined stress in superconducting solenoids from fabrication (winding stress), cooldown (thermal stress), and operation (magnetic stress). Along with a brief discussion of the composite nature of the windings, orthotropic material properties are introduced. The shear stress is again assumed to be zero. A fundamentally two dimensional analysis is formulated for both plane stress and plane strain assumptions for the axial direction. The presence of axial forces is acknowledged and discussed qualitatively. A comparison is made between the two-dimensional analysis and a three-dimensional finite element numerical calculation of an example coil which includes axial forces.

In all of these analytical treatments, the axial stress is either set to zero or introduced in an approximate way through superposition. In a solenoid, the tangential stress and strain are dominant in magnitude and relatively independent of the axial stress, accounting for the relative lack of attention given to the axial stress. As coils become larger, the axial stress becomes more important for magnet design. This is due partly to the stress and strain dependence of high field superconductors. It is also due to the coupling of external reinforcement to the axial stress distribution.

The usual assumption in the preceding analyses is that the average current density is a constant in the section of the solenoid for which the equations are being formulated. This corresponds to the case when a conductor of a given size is wound with a constant amount of insulation, and possibly a constant amount of distributed reinforcement. A more general case would allow a nonuniform amount of distributed reinforcement, resulting in a nonuniform current density in the coil section. In the context of examining such a case, Mitchell and Mszanowski[7] formulate the solenoid stress problem in three dimensions. As in the preceding two-dimensional analyses, simplifications are introduced by assuming zero shear stress and isotropic material properties. The further assumption is made, stated as following the convention for pressure vessels, that the axial strain is constant through the windings. It is further noted that through a radial section of the windings, axial force balance is achieved when the axial stress integrated over that section equals the applied axial load on the section. For the special case of a uniform current density solenoid, the differential equation for the axial stress is given and from the form of the equation and the numerical solutions that were obtained, general aspects of the axial stress distribution in solenoids were inferred. Namely, the axial stress was seen to be nonuniform, with a monotonically decreasing value from the inside of the solenoid outward through the windings, and with the possibility of a positive tension at the bore.

An examination of the axial stress distribution in solenoids, with and without external reinforcement, was made by Markiewicz *et al.*[8] It was noted that, directly from the constitutive equation for the axial strain, the nonuniform distribution of tangential and radial stress in a solenoid will couple through the Poisson ratio to give a nonuniform axial stress. The condition of a constant axial strain through the central region of a solenoid is introduced on the physical grounds that, to a high degree, solenoids remain a right cylinder in operation. The requirement that in the absence of axial loading, the Poisson ratio induced axial stress balances to a zero net axial force leads directly to a value of the constant axial strain.

The assumptions of constant axial strain and zero shear were then shown to give a full three-dimensional solution of the stress balance equations in a GPS analysis.[9] The material properties were assumed fully orthotropic, linear elastic. The solutions for the mechanical stress were presented in general form with constant coefficients. The solution was demonstrated with examples.

From the results, a physical picture of the stress in a solenoid emerges. The tangential stress dominates as a reaction to the radial component of the Lorentz force. The tangential stress is coupled through displacements to the radial stress, which predominantly determine the inplane strains. From the in-plane stress, the Poisson ratio results in an axial stress as well. The Poisson ratio axial stress combines with the axial stress induced by the axial loads, both magnetic and thermal, to give the total axial load. The radial magnetic force loading is relatively uniform in the central region of a solenoid, but decreases toward the end. In a long coil, the axial load originates predominately toward the end of the coil. The radial dependence of the axial load at the end of a coil redistributes to produce a relatively uniform axial load over the central region of the solenoid. In this picture, the influences tending to distort the center of a coil from a straight cylindrical shape, including the nonuniform radial loading at the end of the coil, and the Poisson ratio stress, are taken to be relatively small. The assumption is made that the solenoid remains a right cylinder and this is expressed as a constant value of the axial strain.

The only generality not addressed by the GPS analysis is shear. A formulation was demonstrated by Cox *et al.*[10] in which the shear component of the equations could be retained by employing a power series form for both field and strain. The resulting mathematical formalism expands dramatically in complexity with the inclusion of shear, so much as to perhaps reach a point of diminishing returns between an analytical solution and a numerical solution. Thus, while the possibility of a more general formalism exists, at least in a series approximation, the experience has been that GPS offers a useful and accessible treatment of the three-dimensional stress in solenoids. The derivation of the equations was presented earlier for the mechanical loads.[9] Here, the equations for both mechanical and thermal contraction stress, together with the evaluation of the coefficients for the primary configurations of application are given.

V. GENERALIZED PLANE STRAIN ASSUMPTION

The objective leading to the GPS analysis is to find an analytical solution to the stress balance equations that includes the essential aspects of the axial stress. The simplifying assumption of GPS is introduced on the basis of general physical considerations. The validity of the assumption, and the accuracy of the resulting solution to describe magnetic solenoid configurations of interest, is judged by comparison with numerical solutions.

The dominant loads on a solenoid are the outward radial component and the axial compression component of the Lorentz force. In a long solenoid, the radial force is relatively uniform along the length and the axial force occurs primarily toward the ends of the coil. The primary response of a coil is to expand radially and compress axially. The fundamental assumption of the analysis is that the associated deformation takes the initial right cylinder of the solenoid into a right cylinder in such a way that the displacement vector has components with the functional form

$$\begin{aligned} u &= u(r), \\ w &= w(z). \end{aligned} \tag{8}$$

This deformation maintains the lines of constant r parallel with the z axis, and the planes of constant z normal to the z axis.

A single coil may be constructed as a compound coil with a number of distinct radial sections in contact along common cylindrical boundaries. The radial sections may be distinguished by their material properties and current density. Importantly, in coils with more than one radial section, the above assumption applies to all radial sections uniformly.

For this assumed right cylinder deformation, it is observed from the strain–displacement relations that the shear strain γ_{rz} is zero, and from the form of the material properties that the associated shear stress is zero. It is also seen that the functional dependence of the normal strains is

$$\epsilon_r = \epsilon_r(r),$$
$$\epsilon_\theta = \epsilon_\theta(r), \tag{9}$$
$$\epsilon_z = \epsilon_z(z).$$

Therefore, the assumed right cylinder deformation leads to the condition that at any given axial location the axial strain ϵ_z is a constant as a function of the radius, and that this applies to both simple and compound coils.

With the shear being zero, the second stress balance equation reduces to

$$\frac{\partial \sigma_z}{\partial z} + X_z = 0. \tag{10}$$

Using the stress–strain relations and the functional dependence of the strain, this equation is expressed as

$$C_{zz}\frac{\partial \epsilon_z}{\partial z} + X_z = 0. \tag{11}$$

The further assumption is made that the analysis applies to the central region of a long solenoid. The axial force density X_z in the central region is small, being zero at the midplane, contributes a small amount to the total axial load, and is taken to be zero in the present approximation. The result of the right cylinder deformation and axial force assumption is that the axial strain is constant in the central region:

$$\epsilon_z = \text{constant}. \tag{12}$$

Thus the assumptions imply a state of axial plane strain.

The above discussion is formulated in terms of the mechanical stress and strain associated with magnetic force loads, in recognition of the importance of magnetic loads in solenoid stress. The same discussion can be applied to the thermal case with the conclusion that the axial mechanical strain from thermal loads is constant within a single coil section, and that the total axial strain is constant over all radial sections of a compound coil:

$$\epsilon_z^{\text{tot}} = \text{constant}. \tag{13}$$

Essential to this constant axial strain result is the assumption of a right cylinder deformation. Another description of this assumption is that no bending is induced in the solenoid by the magnetic loads or thermal contraction loads. In fact, there are several ways that a degree of bending is induced. The radial force is not entirely uniform along the length of a solenoid, but typically decreases toward the ends. The radial force is highly nonuniform through the thickness of a coil, falling off nearly linearly with radius, resulting in a nonuniform Poisson ratio stress. The axial force at the end of a coil is not uniform through the thickness of a coil, but typically has a roughly parabolic shape. Also, the axial thermal contraction may differ in different radial sections of a coil. The assumption made here is that the bending associated with each of these conditions, and the associated variation in axial strain, is sufficiently small in comparison with the constant value in the absence of bending as to be seen as a variation about a first order constant value.

VI. STRESS BALANCE EQUATION, GPS APPROXIMATION

From the functional form of the strain, the normal components of the stress are reduced to functions of the radius only. The approximation is made that the axial field B_z, and the associated force density in the windings X_r, are also independent of z and a function of r only. Along with the result that the shear stress is zero, the force balance equation under the assumption of GPS takes the form

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + X_r = 0, \tag{14}$$

where X_r is zero for the case of thermal stress only.

A. Mechanical stress, general solution

The case of Lorentz force loading only is first examined. The thermal contractions in the strain–displacement relations, Eq. (6), are then zero. Combining the strain–displacement with the stress–strain relations, Eq. (1), the stress as a function of displacement is introduced into the stress balance equation to yield

$$C_{rr}\frac{d}{dr}\left(r\frac{du}{dr}\right) - C_{\theta\theta}\frac{u}{r} + (C_{rz} - C_{\theta z})\epsilon_z = -rJ_\theta B_z(r). \tag{15}$$

The assumption is made that the radial distribution of the axial field is linear. This assumption is not fundamental to the analysis and with changes in the following algebra a higher order polynomial form of the field dependence could be adopted. The accuracy of the assumption has been studied in some detail.[5] The axial field of a coil or coil section of constant current density is given by Eq. (16) where the constants B_c and C_0 are determined by the values of B_z at the inside and outside radius:

$$B_z(r) = B_c - C_0 r. \tag{16}$$

Incorporating Eq. (16) in Eq. (15) yields Eq. (17), where the customary variable k is the anisotropy factor given in Eq. (18):

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} - k^2\frac{u}{r} = -\frac{J_\theta B_c}{C_{rr}}r + \frac{J_\theta C_0}{C_{rr}}r^2 - \frac{(C_{rz} - C_{\theta z})}{C_{rr}}\epsilon_z \tag{17}$$

$$k^2 = \frac{C_{\theta\theta}}{C_{rr}}. \tag{18}$$

The general solution of Eq. (17) may be written in the form

$$u = D_1 r^k + D_2 r^{-k} + A_1\epsilon_z r + A_2 r^2 + A_3 r^3, \tag{19}$$

where the known constants are given as

$$A_1 = -\frac{1}{(1-k^2)}\frac{(C_{rz} - C_{\theta z})}{C_{rr}},$$

$$A_2=-\frac{1}{(4-k^2)}\frac{J_\theta B_c}{C_{rr}},\tag{20}$$

$$A_3=\frac{1}{(9-k^2)}\frac{J_\theta C_0}{C_{rr}}.$$

By inspection, there are three singular values of the anisotropy parameter k. When k has the value 1, 2, or 3, the associated constant A is replaced in the general solution by the corresponding function A in Eqs. (21), (22), or (23), respectively:

$$A_1(r)=-\frac{1}{2}\frac{(C_{rz}-C_{\theta z})}{C_{rr}}\ln r\tag{21}$$

$$A_2(r)=-\frac{1}{4}\frac{J_\theta B_c}{C_{rr}}\ln r\tag{22}$$

$$A_3(r)=+\frac{1}{6}\frac{J_\theta C_0}{C_{rr}}\ln r.\tag{23}$$

In the remainder of the analysis, nonsingular values of k are assumed.

The solution for the displacement is used in Eqs. (6) and (1) to yield expressions for the three-dimensional state of stress and strain as follows:

$$\epsilon_\theta=D_1r^{k-1}+D_2r^{-k-1}+A_1\epsilon_z+A_2r+A_3r^2,\tag{24}$$

$$\epsilon_r=kD_1r^{k-1}-kD_2r^{-k-1}+A_1\epsilon_z+2A_2r+3A_3r^2,\tag{25}$$

$$\begin{aligned}\sigma_\theta=&(C_{\theta\theta}+kC_{\theta r})D_1r^{k-1}+(C_{\theta\theta}-kC_{\theta r})D_2r^{-k-1}\\&+(C_{\theta\theta}+C_{\theta r})A_1\epsilon_z+(C_{\theta\theta}+2C_{\theta r})A_2r\\&+(C_{\theta\theta}+3C_{\theta r})A_3r^2+C_{\theta z}\epsilon_z,\end{aligned}\tag{26}$$

$$\begin{aligned}\sigma_r=&(C_{\theta r}+kC_{rr})D_1r^{k-1}+(C_{\theta r}-kC_{rr})D_2r^{-k-1}\\&+(C_{\theta r}+C_{rr})A_1\epsilon_z+(C_{\theta r}+2C_{rr})A_2r\\&+(C_{\theta r}+3C_{rr})A_3r^2+C_{rz}\epsilon_z,\end{aligned}\tag{27}$$

$$\begin{aligned}\sigma_z=&(C_{\theta z}+kC_{rz})D_1r^{k-1}+(C_{\theta z}-kC_{rz})D_2r^{-k-1}\\&+(C_{\theta z}+C_{rz})A_1\epsilon_z+(C_{\theta z}+2C_{rz})A_2r\\&+(C_{\theta z}+3C_{rz})A_3r^2+C_{zz}\epsilon_z.\end{aligned}\tag{28}$$

These equations, together with the value of the coefficients D_1 and D_2, plus the value of the constant strain ϵ_z, determine the distribution of stress and strain in a given coil.

B. Equations for coefficients

The general solution to the stress balance equation may be applied to a coil with a number of distinct, yet mechanically connected radial sections, the sections being differentiated by mechanical properties and current density. The solution for the coefficients D_1 and D_2 results from the boundary conditions associated with each radial coil section. For a stand-alone coil with a single section, the radial stress at the inside and outside radius will be zero. For a coil with several radial sections, additional conditions result from the continuity of the radial stress and radial displacement at the interface between each section. In this way, the number of equations which results is equal to the number of coefficients D. An additional equation is required to determine the unknown value of the axial strain.

Using the concept of a plane through the coil at a given axial location, the static equilibrium of the coil requires that the local product of stress times area accumulated over the plane is equal to the total applied axial load. Thus, on a plane through the coil at an axial position z,

$$\int_{a_1}^{a_{n+1}}2\pi r\sigma_z dr=F_z,\tag{29}$$

where a_1 and a_{n+1} are the inside and outside radii of a coil which has n distinct radial sections, and F_z is the total axial Lorentz force between the plane and the end of the coil over all radial sections. This equation provides the additional condition necessary to determine the axial strain.

1. Coefficients for single section coil

For a single section coil with constant current density and uniform material properties, the boundary conditions are

$$\sigma_r=0\quad\text{at }r=a_1,\tag{30}$$

$$\sigma_r=0\quad\text{at }r=a_2,\tag{31}$$

$$\int_{a_1}^{a_2}2\pi r\sigma_z dr=F_z.\tag{32}$$

Evaluating Eq. (27) at the inner radius results in

$$a_{11}D_1+a_{12}D_2+a_{13}\epsilon_z=b_1,\tag{33}$$

where the constants are given by

$$\begin{aligned}a_{11}&=(C_{\theta r}+kC_{rr})a_1^{k-1},\\a_{12}&=(C_{\theta r}-kC_{rr})a_1^{-k-1},\\a_{13}&=(C_{\theta r}+C_{rr})A_1+C_{rz},\\b_1&=-(C_{\theta r}+2C_{rr})A_2a_1-(C_{\theta r}+3C_{rr})A_3a_1^2.\end{aligned}\tag{34}$$

Evaluating Eq. (27) at the outer radius results in

$$a_{21}D_1+a_{22}D_2+a_{23}\epsilon_z=b_2,\tag{35}$$

where the constants are given by

$$a_{21}=(C_{\theta r}+kC_{rr})a_2^{k-1},$$

$$a_{22}=(C_{\theta r}-kC_{rr})a_2^{-k-1},$$
$$a_{23}=a_{13},\tag{36}$$
$$b_2=-(C_{\theta r}+2C_{rr})A_2a_2-(C_{\theta r}+3C_{rr})A_3a_2^2.$$

Integrating Eq. (28) in Eq. (32) results in

$$a_{31}D_1+a_{32}D_2+a_{33}\epsilon_z=b_3,\tag{37}$$

where the constants are given by

$$\begin{aligned}
a_{31}&=(C_{\theta z}+kC_{rz})\frac{a_2^{k+1}-a_1^{k+1}}{k+1},\\
a_{32}&=(C_{\theta z}-kC_{rz})\frac{a_2^{-k+1}-a_1^{-k+1}}{-k+1},\\
a_{33}&=[(C_{\theta z}+C_{rz})A_1+C_{zz}]\frac{a_2^2-a_1^2}{2},\\
b_3&=\frac{F_z}{2\pi}-(C_{\theta z}+2C_{rz})A_2\frac{a_2^3-a_1^3}{3}\\
&\quad-(C_{\theta z}+3C_{rz})A_3\frac{a_2^4-a_1^4}{4}.
\end{aligned}\tag{38}$$

The set of linear Eqs. (33), (35), and (37) may be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix} D_1\\ D_2\\ \epsilon_z\end{bmatrix}=\begin{bmatrix} b_1\\ b_2\\ b_3\end{bmatrix}\tag{39}$$

which is solved in the standard way as

$$\begin{bmatrix} D_1\\ D_2\\ \epsilon_z\end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}^{-1}\begin{bmatrix} b_1\\ b_2\\ b_3\end{bmatrix}.\tag{40}$$

2. Coefficients for single section coil with external reinforcement

For a single section coil with reinforcement, the boundary conditions are applied to the coil section and the reinforcement.

$$\sigma_r^{(1)}=0 \quad \text{at } r=a_1,\tag{41}$$
$$\sigma_r^{(1)}=\sigma_r^{(2)} \quad \text{at } r=a_2,\tag{42}$$
$$u_r^{(1)}=u_r^{(2)} \quad \text{at } r=a_2,\tag{43}$$
$$\sigma_r^{(2)}=0 \quad \text{at } r=a_3.\tag{44}$$

The axial force equilibrium applies to the coil and reinforcement

$$\int_{a_1}^{a_3}2\pi r\sigma_z dr=F_z.\tag{45}$$

Evaluating Eq. (27) at the inner radius results in

$$a_{11}D_1+a_{12}D_2+a_{15}\epsilon_z=b_1,\tag{46}$$

where the constants are given by

$$\begin{aligned}
a_{11}&=(C_{\theta r}+kC_{rr})a_1^{k-1},\\
a_{12}&=(C_{\theta r}-kC_{rr})a_1^{-k-1},\\
a_{15}&=(C_{\theta r}+C_{rr})A_1+C_{rz},\\
b_1&=-(C_{\theta r}+2C_{rr})A_2a_1-(C_{\theta r}+3C_{rr})A_3a_1^2.
\end{aligned}\tag{47}$$

Evaluating Eq. (27) at the interface between the coil and the reinforcement results in

$$a_{21}D_1+a_{22}D_2+a_{23}D_1'+a_{24}D_2'+a_{25}\epsilon_z=b_2,\tag{48}$$

where the constants are given by

$$\begin{aligned}
a_{21}&=(C_{\theta r}+kC_{rr})a_2^{k-1},\\
a_{22}&=(C_{\theta r}-kC_{rr})a_2^{-k-1},\\
a_{23}&=-(C'_{\theta r}+k'C'_{rr})a_2^{k'-1},\\
a_{24}&=-(C'_{\theta r}-k'C'_{rr})a_2^{-k'-1},\\
a_{25}&=[(C_{\theta r}+C_{rr})A_1+C_{rz}]-[(C'_{\theta r}+C'_{rr})A'_1+C'_{rz}],\\
b_2&=-(C_{\theta r}+2C_{rr})A_2a_2-(C_{\theta r}+3C_{rr})A_3a_2^2
\end{aligned}\tag{49}$$

and where the unprimed quantities refer to the coil and the primed quantities refer to the reinforcement.

Evaluating Eq. (19) for the displacement at the interface results in

$$a_{31}D_1+a_{32}D_2+a_{33}D_1'+a_{34}D_2'+a_{35}\epsilon_z=b_3,\tag{50}$$

where the constants are given by

$$\begin{aligned}
a_{31}&=a_2^{k},\\
a_{32}&=a_2^{-k},\\
a_{33}&=-a_2^{k'},\\
a_{34}&=-a_2^{-k'},\\
a_{35}&=A_1a_2-A'_1a_2,\\
b_3&=-A_2a_2^2-A_3a_2^3.
\end{aligned}\tag{51}$$

Evaluating Eq. (27) at the outside radius of the reinforcement results in

$$a_{43}D_1'+a_{44}D_2'+a_{45}\epsilon_z=0,\tag{52}$$

where the constants are given by

$$\begin{aligned}
a_{43}&=(C'_{\theta r}+k'C'_{rr})a_3^{k'-1},\\
a_{44}&=(C'_{\theta r}-k'C'_{rr})a_3^{-k'-1},\\
a_{45}&=(C'_{\theta r}+C'_{rr})A'_1+C'_{rz}.
\end{aligned}\tag{53}$$

Integrating Eq. (28) through coil and reinforcement in Eq. (45) results in

$$a_{51}D_1+a_{52}D_2+a_{53}D_1'+a_{54}D_2'+a_{55}\epsilon_z=b_5,\tag{54}$$

where the constants are given by

$$a_{51}=(C_{\theta z}+kC_{rz})\frac{a_2^{k+1}-a_1^{k+1}}{k+1},$$
$$a_{52}=(C_{\theta z}-kC_{rz})\frac{a_2^{-k+1}-a_1^{-k+1}}{-k+1},$$
$$a_{53}=(C'_{\theta z}+k'C'_{rz})\frac{a_3^{k'+1}-a_2^{k'+1}}{k'+1},$$
$$a_{54}=(C'_{\theta z}-k'C'_{rz})\frac{a_3^{-k'+1}-a_2^{-k'+1}}{-k'+1}, \tag{55}$$
$$a_{55}=[(C_{\theta z}+C_{rz})A_1+C_{zz}]\frac{a_2^2-a_1^2}{2}+[(C'_{\theta z}+C'_{rz})A'_1+C'_{zz}]\frac{a_3^2-a_2^2}{2},$$
$$b_5=-(C_{\theta z}+2C_{rz})A_2\frac{a_2^3-a_1^3}{3}-(C_{\theta z}+3C_{rz})A_3\frac{a_2^4-a_1^4}{4}+\frac{F_z}{2\pi}.$$

The set of linear Eqs. (46), (48), (50), (52), and (54) may be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35}\\ 0 & 0 & a_{43} & a_{44} & a_{45}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}\end{bmatrix}\begin{bmatrix} D_1\\ D_2\\ D'_1\\ D'_2\\ \epsilon_z\end{bmatrix}=\begin{bmatrix} b_1\\ b_2\\ b_3\\ b_4\\ b_5\end{bmatrix} \tag{56}$$

which is solved in the standard way as

$$\begin{bmatrix} D_1\\ D_2\\ D'_1\\ D'_2\\ \epsilon_z\end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35}\\ 0 & 0 & a_{43} & a_{44} & a_{45}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}\end{bmatrix}^{-1}\begin{bmatrix} b_1\\ b_2\\ b_3\\ b_4\\ b_5\end{bmatrix}. \tag{57}$$

3. Coefficients for a general multisection coil

For a multisection coil, in which any section may be a reinforcement region, the boundary conditions are applied to each section.

$$\sigma_r^{(1)}=0 \quad \text{at } r=a_1 \tag{58}$$

$$\sigma_r^{(j)}=\sigma_r^{(j+1)} \quad \text{at } r=a_{j+1};\quad j=1,\quad n-1 \tag{59}$$

$$u_r^{(j)}=u_r^{(j+1)} \quad \text{at } r=a_{j+1};\ j=1,\quad n-1 \tag{60}$$

$$\sigma_r^{(n)}=0 \quad \text{at } r=a_{n+1} \tag{61}$$

The axial force equilibrium applies to all coil sections:

$$\int_{a_1}^{a_{n+1}} 2\pi r\sigma_z dr=F_z. \tag{62}$$

The derivation of the equations for the coefficients proceeds as in the previous cases. The coefficient matrix for the system of $2n+1$ equations and unknowns is given by

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & \cdots & & & a_{12n+1}\\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & \cdots & & & a_{22n+1}\\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & \cdots & & & a_{32n+1}\\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} & 0 & \cdots & & & a_{42n+1}\\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} & 0 & \cdots & & & a_{52n+1}\\ \vdots & \vdots & & & & & & & & & \vdots\\ 0 & 0 & \cdots & & & & & 0 & a_{2n2n-1} & a_{2n2n} & a_{2n2n+1}\\ a_{2n+11} & a_{2n+12} & \cdots & & & & & & & a_{2n+12n} & a_{2n+12n+1}\end{bmatrix} \tag{63}$$

with the solution, as previously, by matrix inversion.

C. Thermal stress, general solution

The case of thermal stress proceeds in a manner similar to the mechanical stress. The distributed mechanical load X_r is zero in the force balance equation, Eq. (14). The strain displacement, Eq. (6) and stress–strain relations, Eq. (1) are combined, and the resulting stress as a function of displacement, thermal contraction, and the constant total axial strain is introduced into the stress balance equation to yield

$$r\frac{d^2u}{dr^2}+\frac{du}{dr}-k^2\frac{u}{r}=-\frac{C_{rz}-C_{\theta z}}{C_{rr}}\epsilon_z^{\text{tot}}+\frac{C_{\theta r}-C_{\theta\theta}}{C_{rr}}\alpha_\theta\Delta T+\frac{C_{rr}-C_{\theta r}}{C_{rr}}\alpha_r\Delta T+\frac{C_{rz}-C_{\theta z}}{C_{rr}}\alpha_z\Delta T, \tag{64}$$

where the anisotropy factor k is given by

$$k^2=\frac{C_{\theta\theta}}{C_{rr}}. \tag{65}$$

The general solution of Eq. (64) may be written in the form

$$u=D_1r^k+D_2r^{-k}+A_1\epsilon_z^{\text{tot}}r+A_2r, \tag{66}$$

where the known constants are given as

$$\begin{aligned}A_1&=-\frac{1}{1-k^2}\frac{C_{rz}-C_{\theta z}}{C_{rr}},\\ A_2&=\frac{1}{1-k^2}\frac{(C_{\theta r}-C_{\theta\theta})\alpha_\theta\Delta T+(C_{rr}-C_{\theta r})\alpha_r\Delta T+(C_{rz}-C_{\theta z})\alpha_z\Delta T}{C_{rr}}.\end{aligned} \tag{67}$$

The solution is singular when k equals one. In this case the constants A in Eq. (67) are replaced in the general solution by the functions A in Eq. (68):

$$\begin{aligned}A_1(r)&=-\frac{1}{2}\frac{C_{rz}-C_{\theta z}}{C_{rr}}\ln(r)\\ A_2(r)&=\frac{1}{2}\frac{(C_{\theta r}-C_{\theta\theta})\alpha_\theta\Delta T+(C_{rr}-C_{\theta r})\alpha_r\Delta T+(C_{rz}-C_{\theta z})\alpha_z\Delta T}{C_{rr}}\ln(r).\end{aligned} \tag{68}$$

The solution for the displacement is introduced through Eqs. (1) and (6) to yield the three-dimensional state of stress and strain given in Eqs. (69)–(73), together with Eq. (13):

$$\epsilon_\theta=D_1r^{k-1}+D_2r^{-k-1}+A_1\epsilon_z^{\text{tot}}+A_2-\alpha_\theta\Delta T \tag{69}$$

$$\epsilon_r=D_1kr^{k-1}-D_2kr^{-k-1}+A_1\epsilon_z^{\text{tot}}+A_2-\alpha_r\Delta T \tag{70}$$

$$\begin{aligned}\sigma_\theta=&(C_{\theta\theta}+kC_{\theta r})D_1r^{k-1}+(C_{\theta\theta}-kC_{\theta r})D_2r^{-k-1}\\&+(C_{\theta\theta}+C_{\theta r})A_1\epsilon_z^{tot}+(C_{\theta\theta}+C_{\theta r})A_2+C_{\theta z}\epsilon_z^{\text{tot}}\\&-C_{\theta\theta}\alpha_\theta\Delta T-C_{\theta r}\alpha_r\Delta T-C_{\theta z}\alpha_z\Delta T\end{aligned} \tag{71}$$

$$\begin{aligned}\sigma_r=&(C_{\theta r}+kC_{rr})D_1r^{k-1}+(C_{\theta r}-kC_{rr})D_2r^{-k-1}\\&+(C_{\theta r}+C_{rr})A_1\epsilon_z^{\text{tot}}+(C_{\theta r}+C_{rr})A_2+C_{rz}\epsilon_z^{\text{tot}}\\&-C_{\theta r}\alpha_\theta\Delta T-C_{rr}\alpha_r\Delta T-C_{rz}\alpha_z\Delta T\end{aligned} \tag{72}$$

$$\begin{aligned}\sigma_z=&(C_{\theta z}+kC_{rz})D_1r^{k-1}+(C_{\theta z}-kC_{rz})D_2r^{-k-1}\\&+(C_{\theta z}+C_{rz})A_1\epsilon_z^{\text{tot}}+(C_{\theta z}+C_{rz})A_2+C_{zz}\epsilon_z^{\text{tot}}\\&-C_{\theta z}\alpha_\theta\Delta T-C_{rz}\alpha_r\Delta T-C_{zz}\alpha_z\Delta T.\end{aligned} \tag{73}$$

These equations, together with the value of the coefficients D_1 and D_2, plus the value of the constant strain ϵ_z^{tot}, determine the stress and strain in the coil section.

D. Equations for coefficients

The solution of equations for the coefficients proceeds essentially in the same manner as the case of magnetic loads, but here with no applied axial magnetic load, the net internal axial stress in the central region of a coil must balance to zero.

1. Coefficients for single section coil

For a single section coil with constant current density and uniform material properties, the boundary conditions are

$$\sigma_r=0 \quad \text{at } r=a_1 \tag{74}$$

$$\sigma_r=0 \quad \text{at } r=a_2 \tag{75}$$

$$\int_{a_1}^{a_2}2\pi r\sigma_z dr=0. \tag{76}$$

Evaluating Eq. (72) at the inner radius results in

$$a_{11}D_1+a_{12}D_2+a_{13}\epsilon_z^{\text{tot}}=b_1, \tag{77}$$

where the constants are given by

$$\begin{aligned}a_{11}&=(C_{\theta r}+kC_{rr})a_1^{k-1},\\ a_{12}&=(C_{\theta r}-kC_{rr})a_1^{-k-1},\\ a_{13}&=(C_{\theta r}+C_{rr})A_1+C_{rz},\\ b_1&=-(C_{\theta r}+C_{rr})A_2+C_{\theta r}\alpha_\theta\Delta T+C_{rr}\alpha_r\Delta T+C_{rz}\alpha_z\Delta T.\end{aligned} \tag{78}$$

Evaluating Eq. (72) at the outer radius results in

$$a_{21}D_1+a_{22}D_2+a_{23}\epsilon_z^{\text{tot}}=b_2, \tag{79}$$

where the constants are given by

$$a_{21}=(C_{\theta r}+kC_{rr})a_2^{k-1},$$

$$a_{22}=(C_{\theta r}-kC_{rr})a_2^{-k-1},\quad a_{23}=a_{13},\quad b_2=b_1. \tag{80}$$

Integrating Eq. (73) in Eq. (76) results in

$$a_{31}D_1+a_{32}D_2+a_{33}\epsilon_z^{\text{tot}}=b_3, \tag{81}$$

where the constants are given by

$$\begin{aligned}
a_{31}&=(C_{\theta z}+kC_{rz})\frac{a_2^{k+1}-a_1^{k+1}}{k+1},\\
a_{32}&=(C_{\theta z}-kC_{rz})\frac{a_2^{-k+1}-a_1^{-k+1}}{-k+1},\\
a_{33}&=[(C_{\theta z}+C_{rz})A_1+C_{zz}]\frac{a_2^2-a_1^2}{2},\\
b_3&=-[(C_{\theta z}+C_{rz})A_2-C_{\theta z}\alpha_\theta\Delta T-C_{rz}\alpha_r\Delta T\\
&\quad -C_{zz}\alpha_z\Delta T]\frac{a_2^2-a_1^2}{2}.
\end{aligned} \tag{82}$$

The set of linear Eqs. (77), (79), and (81) may be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix} D_1\\ D_2\\ \epsilon_z^{\text{tot}}\end{bmatrix}=\begin{bmatrix} b_1\\ b_2\\ b_3\end{bmatrix} \tag{83}$$

which is solved in the standard way as

$$\begin{bmatrix} D_1\\ D_2\\ \epsilon_z^{\text{tot}}\end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}^{-1}\begin{bmatrix} b_1\\ b_2\\ b_3\end{bmatrix}. \tag{84}$$

2. Coefficients for single section coil with external reinforcement

For a single section coil with reinforcement, the boundary conditions are applied to the coil section and the reinforcement.

$$\sigma_r^{(1)}=0 \quad \text{at } r=a_1 \tag{85}$$

$$\sigma_r^{(1)}=\sigma_r^{(2)} \quad \text{at } r=a_2 \tag{86}$$

$$u_r^{(1)}=u_r^{(2)} \quad \text{at } r=a_2 \tag{87}$$

$$\sigma_r^{(2)}=0 \quad \text{at } r=a_3 \tag{88}$$

The axial force equilibrium applies to the coil and reinforcement.

$$\int_{a_1}^{a_3}2\pi r\sigma_z dr=0 \tag{89}$$

Evaluating Eq. (72) at the inner radius results in

$$a_{11}D_1+a_{12}D_2+a_{15}\epsilon_z^{\text{tot}}=b_1, \tag{90}$$

where the constants are given by

$$\begin{aligned}
a_{11}&=(C_{\theta r}+kC_{rr})a_1^{k-1},\\
a_{12}&=(C_{\theta r}-kC_{rr})a_1^{-k-1},\\
a_{15}&=(C_{\theta r}+C_{rr})A_1+C_{rz},\\
b_1&=-(C_{\theta r}+C_{rr})A_2+C_{\theta r}\alpha_\theta\Delta T+C_{rr}\alpha_r\Delta T+C_{rz}\alpha_z\Delta T.
\end{aligned} \tag{91}$$

Evaluating Eq. (72) at the interface between the coil and the reinforcement results in

$$a_{21}D_1+a_{22}D_2+a_{23}D_1'+a_{24}D_2'+a_{25}\epsilon_z^{\text{tot}}=b_2, \tag{92}$$

where the constants are given by

$$\begin{aligned}
a_{21}&=(C_{\theta r}+kC_{rr})a_2^{k-1},\\
a_{22}&=(C_{\theta r}-kC_{rr})a_2^{-k-1},\\
a_{23}&=-(C_{\theta r}'+k'C_{rr}')a_2^{k'-1},\\
a_{24}&=-(C_{\theta r}'-k'C_{rr}')a_2^{-k'-1},\\
a_{25}&=[(C_{\theta r}+C_{rr})A_1+C_{rz}]-[(C_{\theta r}'+C_{rr}')A_1'+C_{rz}'],\\
b_2&=[(C_{\theta r}'+C_{rr}')A_2'-C_{\theta r}'\alpha_\theta'\Delta T-C_{rr}'\alpha_r'\Delta T-C_{rz}'\alpha_z'\Delta T]\\
&\quad -[(C_{\theta r}+C_{rr})A_2-C_{\theta r}\alpha_\theta\Delta T-C_{rr}\alpha_r\Delta T-C_{rz}\alpha_z\Delta T],
\end{aligned} \tag{93}$$

and where the unprimed quantities refer to the coil and the primed quantities refer to the reinforcement.

Evaluating Eq. (66) for the displacement at the interface results in

$$a_{31}D_1+a_{32}D_2+a_{33}D_1'+a_{34}D_2'+a_{35}\epsilon_z^{\text{tot}}=b_3, \tag{94}$$

where the constants are given by

$$\begin{aligned}
&a_{31}=a_2^k \quad a_{32}=a_2^{-k},\\
&a_{33}=-a_2^{k'} \quad a_{34}=-a_2^{-k'},\\
&a_{35}=A_1a_2-A_1'a_2 \quad b_3=A_2'a_2-A_2a_2.
\end{aligned} \tag{95}$$

Evaluating Eq. (72) at the outside radius of the reinforcement results in

$$a_{43}D_1'+a_{44}D_2'+a_{45}\epsilon_z^{\text{tot}}=b_4, \tag{96}$$

where the constants are given by

$$\begin{aligned}
a_{43}&=(C_{\theta r}'+k'C_{rr}')a_3^{k'-1},\\
a_{44}&=(C_{\theta r}'-k'C_{rr}')a_3^{-k'-1},\\
a_{45}&=(C_{\theta r}'+C_{rr}')A_1'+C_{rz}',\\
b_4&=-(C_{\theta r}'+C_{rr}')A_2'+C_{\theta r}'\alpha_\theta'\Delta T+C_{rr}'\alpha_r'\Delta T+C_{rz}'\alpha_z'\Delta T.
\end{aligned} \tag{97}$$

Integrating Eq. (73) through coil and reinforcement in Eq. (89) results in

$$a_{51}D_1+a_{52}D_2+a_{53}D_1'+a_{54}D_2'+a_{55}\epsilon_z^{\text{tot}}=b_5, \tag{98}$$

where the constants are given by

$$a_{51}=(C_{\theta z}+kC_{rz})\frac{a_2^{k+1}-a_1^{k+1}}{k+1},$$
$$a_{52}=(C_{\theta z}-kC_{rz})\frac{a_2^{-k+1}-a_1^{-k+1}}{-k+1},$$
$$a_{53}=(C'_{\theta z}+k'C'_{rz})\frac{a_3^{k'+1}-a_2^{k'+1}}{k'+1},$$
$$a_{54}=(C'_{\theta z}-k'C'_{rz})\frac{a_3^{-k'+1}-a_2^{-k'+1}}{-k'+1}, \tag{99}$$
$$a_{55}=[(C_{\theta z}+C_{rz})A_1+C_{zz}]\frac{a_2^2-a_1^2}{2}+[(C'_{\theta z}+C'_{rz})A'_1+C'_{zz}]\frac{a_3^2-a_2^2}{2},$$
$$b_5=-[(C_{\theta z}+C_{rz})A_2-C_{\theta z}\alpha_\theta\Delta T-C_{rz}\alpha_r\Delta T-C_{zz}\alpha_z\Delta T]\frac{a_2^2-a_1^2}{2}-[(C'_{\theta z}+C'_{rz})A'_2-C'_{\theta z}\alpha'_\theta\Delta T-C'_{rz}\alpha'_r\Delta T-C'_{zz}\alpha'_z\Delta T]\frac{a_3^2-a_2^2}{2}.$$

The set of linear Eqs. (90), (92), (94), (96), and (98) may be written in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35}\\ 0 & 0 & a_{43} & a_{44} & a_{45}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}\end{bmatrix}\begin{bmatrix} D_1\\ D_2\\ D'_1\\ D'_2\\ \epsilon_z^{\text{tot}}\end{bmatrix}=\begin{bmatrix} b_1\\ b_2\\ b_3\\ b_4\\ b_5\end{bmatrix} \tag{100}$$

which is solved in the standard way as

$$\begin{bmatrix} D_1\\ D_2\\ D'_1\\ D'_2\\ \epsilon_z^{\text{tot}}\end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35}\\ 0 & 0 & a_{43} & a_{44} & a_{45}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}\end{bmatrix}^{-1}\begin{bmatrix} b_1\\ b_2\\ b_3\\ b_4\\ b_5\end{bmatrix} \tag{101}$$

3. Coefficients for a general multisection coil

For a multisection coil, in which any section may be a reinforcement region, the boundary conditions are applied to each section.

$$\sigma_r^{(1)}=0 \quad \text{at } r=a_1 \tag{102}$$

TABLE I. Magnet parameters of a simple solenoid with external reinforcement.

a_1 (mm)	a_2 (mm)	a_3 (mm)	$h/2$ (mm)	J (A/mm^2)
250	300	310	500	150

TABLE II. Material properties assumed for finite element and generalized plane strain analysis.

Property	Coil	Reinforcement
E_θ(GPa)	95	190
E_r(GPa)	50	125
E_z(GPa)	60	140
G_{rz}(GPa)	12	35
$\nu_{\theta r}$	0.330	0.275
ν_{rz}	0.200	0.190
$\nu_{\theta z}$	0.330	0.275
$\nu_{r\theta}$	0.174	0.181
ν_{zr}	0.240	0.213
$\nu_{z\theta}$	0.208	0.203
$\alpha_\theta\Delta T$	−0.003 15	−0.003 00
$\alpha_r\Delta T$	−0.004 50	−0.003 30
$\alpha_z\Delta T$	−0.003 70	−0.003 20

$$\sigma_r^{(j)}=\sigma_r^{(j+1)} \quad \text{at } r=a_{j+1};\ j=1, \quad n-1 \tag{103}$$
$$u_r^{(j)}=u_r^{(j+1)} \quad \text{at } r=a_{j+1};\ j=1, \quad n-1 \tag{104}$$
$$\sigma_r^{(n)}=0 \quad \text{at } r=a_{n+1} \tag{105}$$

The axial force equilibrium applies to all coil sections.

$$\int_{a_1}^{a_{n+1}} 2\pi r\sigma_z dr=0 \tag{106}$$

The derivation of the equations for the coefficients proceeds as in the previous cases. The coefficient matrix has the form given by Eq. (63).

VII. EXAMPLE CALCULATIONS

The results of a GPS calculation are compared with a finite element calculation for a superconducting magnet consisting of a single solenoid coil with external reinforcement. The parameters of the coil are given in Table I, where a_1 and a_2 are the inside and outside radius of the windings, respectively, a_2 and a_3 are the inside and outside radius of the reinforcement, and h is the length. The current density J is the average value over the winding pack, including conductor and insulation. The material properties given in Table II are similarly average properties over the windings and over the reinforcement region. The properties are characteristic of an epoxy impregnated, wire wound coil construction.

The largest loads on a superconducting magnet are from the magnetic force. The axial, radial, and tangential components of stress are given in Fig. 1. The tangential stress, which reacts the radial outward component of the force, is the dominant stress component. In this example, the conductor region of the windings is supported by the external reinforcement, which displays an increased stress in proportion to E_θ. The axial stress is also significant in magnitude. The value of the axial stress in the reinforcement is an indication of the proportion of the axial load that is supported by the reinforcement. The corresponding strain components for the magnetic force load are given in Fig. 2.

The thermal stress, which develops as a result of cooldown from room temperature to liquid helium operating temperature, is shown in Fig. 3. Both the axial and tangential

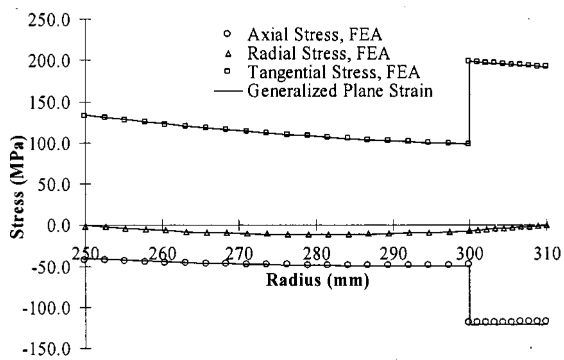

FIG. 1. Axial, radial, and tangential stress components due to magnetic force loading.

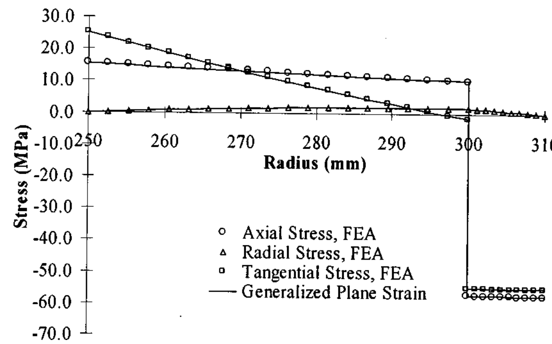

FIG. 3. Axial, radial, and tangential stress components due to differential thermal contraction during cooldown.

stress distributions represent a zero net force balance in the coil. The thermal strain components are given in Fig. 4. The tangential strain reflects the nonisotropic nature of the thermal contraction, with the inside and the outside of the coil being drawn toward the center.

The correspondence between the finite element and GPS calculations may be assessed from the figures. The major simplifying assumption of GPS is the assumption of a constant axial strain. The validity of the assumption for this particular example is shown with greater resolution in Figs. 5 and 6, for the magnetic force and thermal contraction, respectively. In both cases, it is the total axial strain that is shown, which for the case of thermal contraction is the sum of the mechanical strain resulting from the thermal contraction loads and the thermal contraction strain. For the case of magnetic force, the degree of bending in a coil and the deviation from a constant value of the axial strain, will depend in general on all the coils in a set of coils which produce a magnetic field at the coil in question.

A constant value of axial strain is equivalent to a right cylinder deformation. Nonconstant values as shown in the finite element results indicate that some degree of bending does occur. Although small, potential sources of bending were examined to gain further insight. In order to identify the source of the observed bending, the same example coil was used with the radial and axial force distributions applied separately. The results are given in Fig. 7. The radial force, being produced by the axial field, is relatively constant over the central region of a long solenoid, but decreases toward

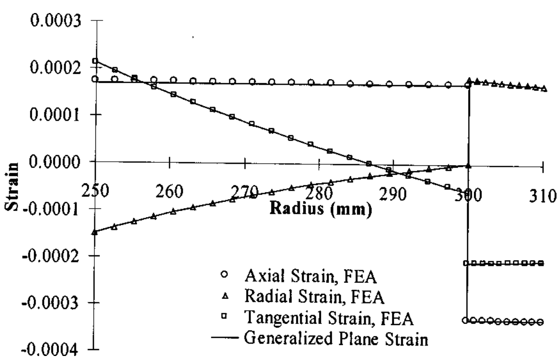

FIG. 4. Axial, radial, and tangential strain components due to differential thermal contraction during cooldown.

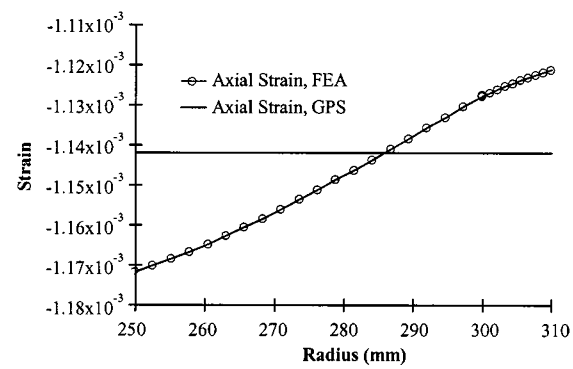

FIG. 5. Total axial strain comparison for magnetic force loading.

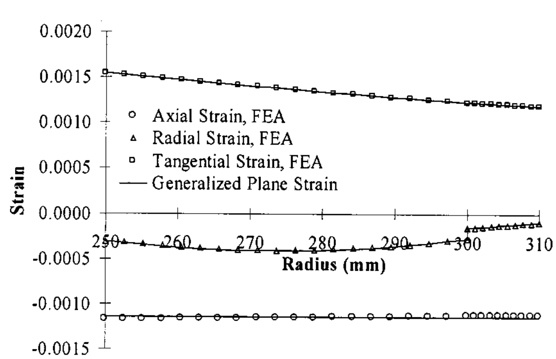

FIG. 2. Axial, radial, and tangential strain components due to magnetic force loading.

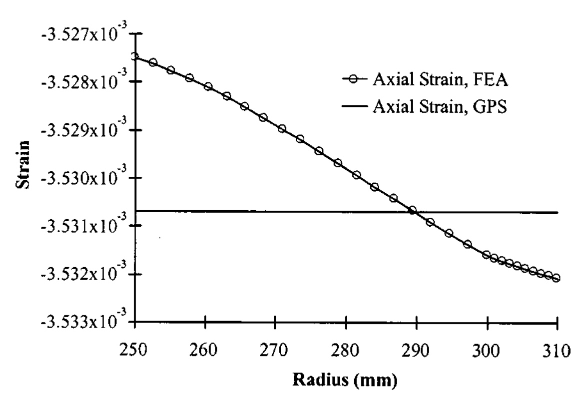

FIG. 6. Total axial strain comparison for thermal contraction loading.

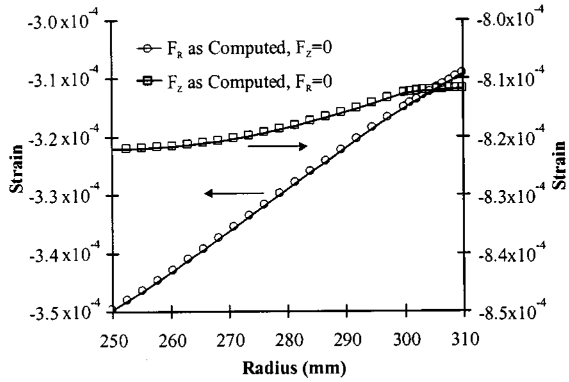

FIG. 7. Total axial strain computed by finite elements for radial magnetic loading and axial magnetic loading separately.

the end. The result is a small tendency of a coil to form a barrel shape. The axial force is distributed only over the windings, while the reinforcement forms a stiff outer shell. The result is an additional degree of bending.

VIII. CONCLUSIONS

The method of GPS, with the assumption of constant axial strain and the integral constraint on the axial stress, provides an accurate analytic method for the calculation of three-dimensional stress in long solenoids. Knowledge of the axial stress and the redistribution of the axial stress into external reinforcement is of particular interest in coils of increasing size and field strength. The analysis has served to focus on the extent to which the axial strain in solenoid coils is a constant, and to examine the relationship between the condition of zero shear stress and the constant axial strain condition.

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