Power Series Stress Analysis of Solenoid Magnets

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Abstract—An analytical solution for the stress analysis of superconducting solenoids is given which reflects the full spatial variation of Lorentz body forces and includes shear stress components. The solution is based on power series expansions of the Lorentz body forces and the displacements which in turn gives the strains and stresses, including shear stress components. This solution is applicable to the windings and reinforcement layers of magnets and is compared to a finite element method using a 7 T sample magnet with reinforcement.

I. Introduction

An analytical formulation of the stress analysis of superconducting solenoids is presented which is applicable throughout the windings of a coil, and includes the shear stress in addition to the normal stress components. As the size and field strength of high field solenoids increase, knowledge of the stress distribution beyond the central region of the windings becomes increasingly important. This is especially true for coils with external reinforcement, where the maximum shear stress between the coil and reinforcement occurs toward the end of the coil. Previous analytical solutions provide the stress distribution near the mid-plane of a solenoid, neglecting shear.

Early analyses of stress at the mid-plane of a solenoid were limited to tangential and radial stress. A plane stress analysis was presented including external reinforcement [1]. Analytical solutions were extended to include plain stress and plain strain assumptions, and were applied to thermal and winding stress in addition to magnetic stress [2]. The extension of the analysis to include axial stress was made in the generalized plain strain solution [3]. The basic assumption in all these analyses was zero shear stress, limiting the applicability of the solutions to a neighborhood of the mid-plane. A set of equates has been presented previously to treat the three dimensional stress distribution generally, including shear stress, based on minimization of strain energy. These equations were solved numerically by finite difference methods [4].

The present results employ a power series expansion to approximate the axial and radial Lorentz force distribution throughout the windings. The force distribution is computed separately and the force components are fit to a power series with characteristic radial dependence for each component. Fundamental to the analysis is the use of a similar power series expansion for the components of the displacement. Equations are formulated and solved for the displacements, from which the components of strain and corresponding stress are derived.

The formulation is applied by way of example to a single coil with external reinforcement. Typical orthotropic material properties are assumed for the windings and reinforcement regions. The results are compared to finite element calculations.

II. Solution Formulation

The windings and reinforcement layers of a superconducting solenoid, as shown in Fig. 1, are treated as homogeneous, orthotropic linearly-elastic materials. The distributed Lorentz force, X, the body force vector, act on the windings of the magnet. The related stress equilibrium equations can be derived in cylindrical coordinates as [5]

$$
\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + X_r = 0 \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + X_\theta = 0 \qquad (1) \\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + X_z = 0 .
\end{aligned}
$$

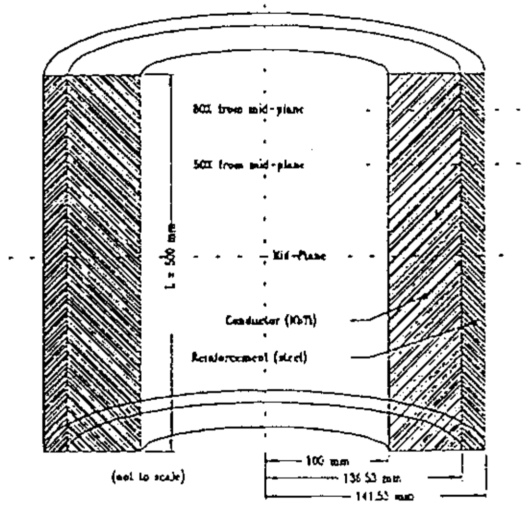

Fig. 1. Single Solenoid with Reinforcement.

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The stress-strain relationships for orthotropic materials can be written in terms of material parameters as

$$\varepsilon_r = \frac{\sigma_r}{E_r} - \nu_{\theta r}\frac{\sigma_\theta}{E_\theta} - \nu_{zr}\frac{\sigma_z}{E_z} \qquad \varepsilon_{r\theta} = \frac{\sigma_{r\theta}}{G_{r\theta}}$$
$$\varepsilon_\theta = -\nu_{r\theta}\frac{\sigma_r}{E_r} + \frac{\sigma_\theta}{E_\theta} - \nu_{z\theta}\frac{\sigma_z}{E_z} \qquad \varepsilon_{rz} = \frac{\sigma_{rz}}{G_{rz}} \tag{2}$$
$$\varepsilon_z = -\nu_{rz}\frac{\sigma_r}{E_r} - \nu_{\theta z}\frac{\sigma_\theta}{E_\theta} + \frac{\sigma_z}{E_z} \qquad \varepsilon_{\theta z} = \frac{\sigma_{\theta z}}{G_{\theta z}}$$

The compliance matrix from (2) can be inverted into the stiffness matrix found in Hooke's law. From the form of the compliance matrix, Hooke's law for orthotropic materials takes the form

$$\sigma_i = C_{ij}\,\varepsilon_j \quad i,j = 1,2,3. \tag{3}$$
$$\sigma_k = C_{kk}\,\varepsilon_k \quad k = 4,5,6.$$

Strain-displacement relations can be specialized for an axi-symmetric system which has the characteristic conditions

$$\tau_{r\theta} = \tau_{z\theta} = 0 \quad \text{and} \quad \partial/\partial\theta = 0. \tag{4}$$

This allows the displacements to be functions of r and z as

$U_r(r,z)$ = radial displacement
$U_z(r,z)$ = axial displacement (5)
$U_\theta = 0$ = tangential displacement.

The strain-displacement relations are simplified to

$$\varepsilon_r = \frac{\partial U_r}{\partial r}, \quad \varepsilon_\theta = \frac{U_r}{r}, \quad \varepsilon_z = \frac{\partial U_z}{\partial z}$$
$$\varepsilon_{r\theta} = \varepsilon_{\theta r} = \varepsilon_{\theta z} = \varepsilon_{z\theta} = 0 \tag{6}$$
$$\varepsilon_{rz} = \varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}\right)$$

By substitution of (6) into (3), the stresses can be described by functions of displacement

$$\sigma_r = f_r(U_r, U_z), \quad \sigma_\theta = f_\theta(U_r, U_z), \quad \sigma_z = f_z(U_r, U_z). \tag{7}$$

Substituting the stress-displacement relations into the equilibrium equations (1) results in two partial differential equations (PDE) as

$$r^2 U_{r,rr} + k_1 r^2 U_{r,zz} + rU_{r,r} + k_5 U_r + k_2 r^2 U_{z,rz} + k_3 r U_{z,z} + k_4 r^2 X_r = 0$$
$$k_7 r U_{r,rz} + k_8 U_{r,z} + rU_{z,zz} + k_6 r U_{z,rr} + k_6 U_{z,r} + k_9 r X_z = 0 \tag{8}$$

where $k = \dfrac{-C_{22}}{C_{11}}$, $k_1 = \dfrac{G_{rz}}{C_{11}}$, $k_2 = \dfrac{C_{13}+G_{rz}}{C_{11}}$,

$$k_3 = \frac{C_{13}-C_{23}}{C_{11}}, \; k_4 = \frac{1}{C_{11}}, \; k_6 = \frac{G_{rz}}{C_{33}},$$
$$k_7 = \frac{G_{rz}+C_{13}}{C_{33}}, \; k_8 = \frac{G_{rz}+C_{23}}{C_{33}}, \; k_9 = \frac{1}{C_{33}}$$

The distributed Lorentz force density or body force, X, is a function of r and z, and is related to the magnetic field and current density by

$$X_r(r,z) = -J_\theta B_z \quad \text{and} \quad X_z(r,z) = -J_\theta B_r \tag{9}$$

This analysis assumes that these body forces may be described using a power series polynomial form

$$X_r(r,z) = \sum_{n=0}^{\infty} f_{2n}(r)\cdot z^{2n}$$
$$X_z(r,z) = \sum_{n=0}^{\infty} g_{2n+1}(r)\cdot z^{2n+1} \tag{10}$$
$$\text{where} \quad f_{2n}(r) = a^{I}_{2n} + a^{II}_{2n}\, r$$
$$g_{2n+1}(r) = b^{I}_{2n+1} + b^{II}_{2n+1}\, r + b^{III}_{2n+1}\, r^2$$

The radial body force is linear with respect to r and even with respect to z; while the axial body force is quadratic with respect to r and odd with respect to z. The body forces used in this analysis utilized curve-fitting of this form to the eighth order in z.

In order to arrive at a solution to the two PDE's (8), the displacements are also assumed to be described by power series forms with respect to z. In this study, the displacements were expanded to the 8th order in z as

$$U_r(r,z) = \sum_{n=0}^{4} u_{2n}(r)\cdot z^{2n} \tag{11}$$
$$U_z(r,z) = \sum_{n=1}^{4} u_{2n-1}(r)\cdot z^{2n-1}$$

Substitution of these displacements (11) and body forces (10), with an expansion to the 8th order in z, into the PDE's (8) results in nine, simultaneous, second order, ordinary differential equations. The ODE's are of the displacement sub-functions in (11) that are functions of r. For the PDE's to be satisfied for all z, each ODE is equated to as

$$(\text{ode}_0) + (\text{ode}_2)\,z^2 + (\text{ode}_4)\,z^4 + (\text{ode}_6)\,z^6 + (\text{ode}_8)\,z^8 = 0 \tag{12}$$
$$(\text{ode}_1)\,z + (\text{ode}_2)\,z^3 + (\text{ode}_5)\,z^5 + (\text{ode}_7)\,z^7 = 0$$

The general form of the displacement sub-functions will be

$$u_n = \sum_m (h^{I}_{n,m} r^m) + \sum_m (h^{II}_{n,m} r^{2m+k}) + \sum_m (h^{III}_{n,m} r^{2m+1}\ln(r)) \tag{13}$$

The analytic solution for the displacement sub-functions gives the displacements and related strain and stress.

The nine ODE's from (12) are second order and therefore each ODE gives two integration constants which make a total of eighteen constants. For a two layer magnet (winding and reinforcement) there are a total of 36 constants. The integration constants within the solution will be found by applying boundary conditions for each layer within a magnet. Each mechanically independent section of a magnet can be analyzed separately. Boundary conditions are applied at each interface, inner and outer radii, and along the ends of the magnet. At the free surfaces (inner radius of winding, r_1, and outer radius of reinforement, r_3) there are a total of four boundary conditions as

$$\sigma_r(r_1,z)=0\,, \qquad \tau_{rz}(r_1,z)=0 \qquad (14)$$
$$\sigma_r(r_3,z)=0\,, \qquad \tau_{rz}(r_3,z)=0$$

In order for each boundary condition to be satisfied for all z, each coefficient of z must be equated to zero, therefore each boundary condition gives four equations.

At the interface between winding and reinforcement, r_2, there are also a total of four continuity conditions as

$$\sigma_r^A(r_2,z)=\sigma_r^B(r_2,z) \qquad (15)$$
$$\tau_{rz}^A(r_2,z)=\tau_{rz}^B(r_2,z)$$
$$U_r^A(r_2,z)=U_r^B(r_2,z)$$
$$U_z^A(r_2,z)=U_z^B(r_2,z)$$

Again each boundary condition gives four equations except for continuity of axial displacement, U_z, which gives three equations.

Five additional boundary conditions are needed to obtain the 36 required. These five are applied at the end of the coil where there is another free surface. At this end the axial and shear stresses are zero. However, there are not enough integration constants to satisfy these conditions completely. Therefore relaxed boundary conditions are applied which equate the integrated axial and shear stress to zero, physically setting the axial and shear forces to zero.

$$\int_{r_1}^{r_3} r\sigma_z\,dr=0 \qquad (16)$$
$$\int_{r_1}^{r_3} r\tau_{rz}\,dr=0$$

The final three conditions applied at the coil end prescribe the axial stress for the winding at the interface and for the reinforcement at the interface and outer radius to zero as

$$\sigma_z^A(r_2,\tfrac{L}{2})=0 \qquad (17)$$
$$\sigma_z^B(r_2,\tfrac{L}{2})=0$$
$$\sigma_z^B(r_3,\tfrac{L}{2})=0$$

III. Comparative Calculations

A stress analysis was performed for a 7 T example magnet using the present power series analysis and also using a finite element method. The parameters of the 7 T solenoid with external reinforcement are given in Table 1.

TABLE 1
PARAMETERS FOR 7 T MAGNET WITH EXTERNAL REINFORCEMENT

Layer	r_i (mm)	r_o (mm)	t (mm)	L (mm)
conductor (NbTi)	100.00	136.53	36.53	500
reinforcement (steel)	136.53	141.53	5.00	500

The material properties of the windings and reinforcement used in the analysis are given in Table 2. The average current density is 169 A/mm^2.

TABLE 2
MATERIAL PROPERTIES FOR WINDINGS AND REINFORCEMENT

	conductor (NbTi)	reinforcement (steel)
E_r (GPa)	45.5	188.9
E_θ (GPa)	81.5	196.5
E_z (Gpa)	49.4	188.9
G_{rz} (GPa)	12.5	68.1
$\nu_{r\theta}$	0.341	0.300
$\nu_{\theta z}$	0.319	0.300
ν_{rz}	0.182	0.288

The body force field distributions were obtained by computing the field throughout the windings. Fig. 2 and 3 show the axial and radial body force fields respectively. These body force field distributions were curve fit using the polynomial expansions described in (10) for expansions through the 8th order in z.

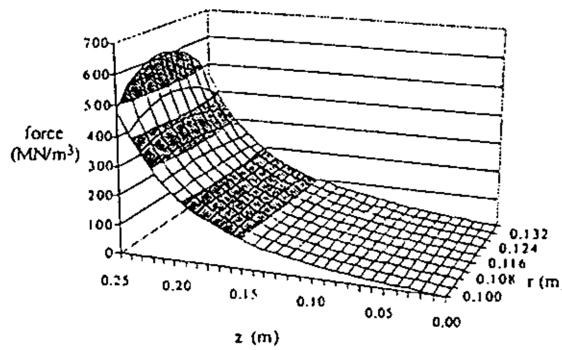

Fig. 2. Axial body force distribution.

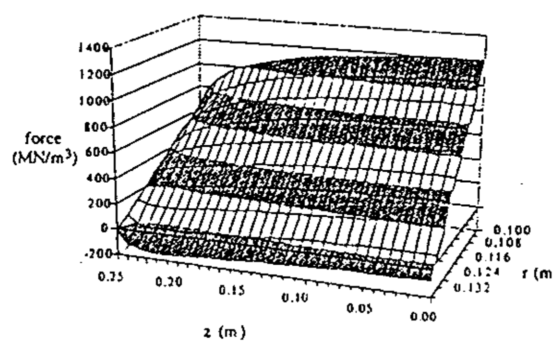

Fig. 3. Radial body force distribution.

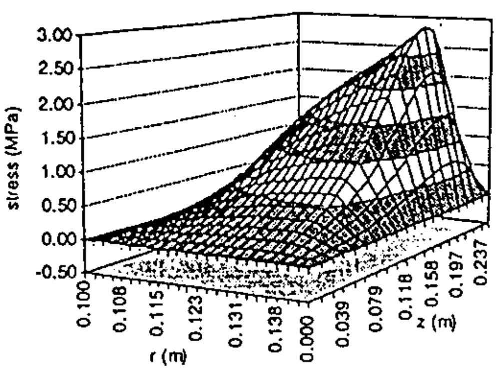

Fig. 4. 3-D Shear stress distribution.

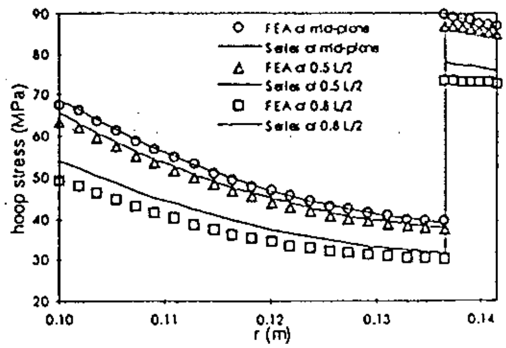

Fig. 5. Comparison of tangential stresses.

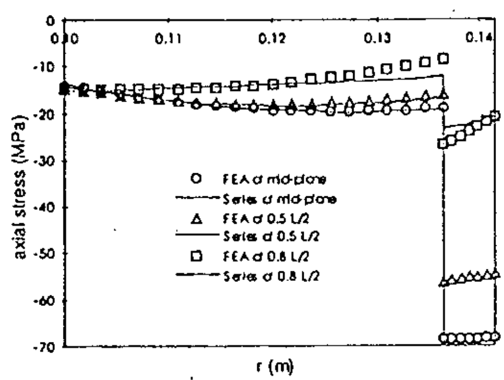

Fig. 6. Comparison of axial stresses.

Fig. 4 gives a 3-D view of the shear stress distribution in order to exhibit the capability of the power series solution to follow 'sharp' curves near the end of the magnet. Fig. 5 through 8 show stress distribution comparisons against finite element method for tangential, axial, radial, and shear stresses. The stress distribution is given as a function of radius throughout the windings and reinforcement, at the mid-plane and at 0.5 and 0.8 of the half-length of the magnet.

IV. Conclusion

An analytical solution to the distribution of stress and strain throughout a solenoid magnet, including the shear

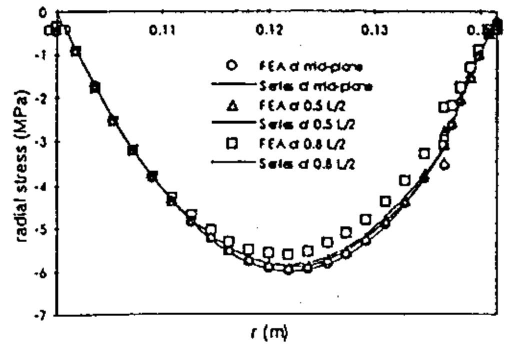

Fig. 7. Comparison of radial stresses.

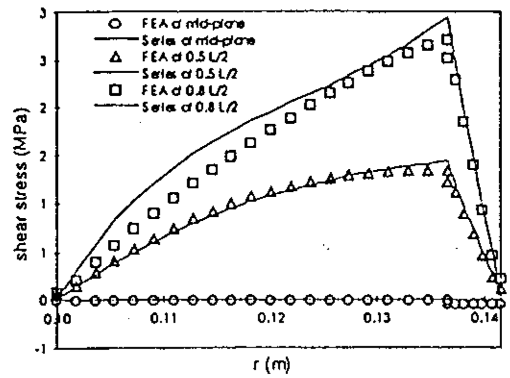

Fig. 8. Comparison of shear stresses.

stress components, has been derived. The solution is based on power series expansions of both the distributed Lorentz body force and the displacements. The analytical solutions were compared with finite element solutions. In order to achieve the degree of correspondence which is shown in the figures, it was found necessary to work with displacement expansions to the 8th order in z. The finite element technique is always available for the detailed analysis of individual magnet designs. An analytical technique such as the subject solution, however, can be conventiently formulated as a part of more general computer codes and used in a powerful manner in the search for designs which optimize a number of criteria, including limitations on the various components of the stress.

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