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Generalized Plane Strain Analysis of Solenoid Magnets

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***Abstract*—A stress analysis assuming generalized plane strain conditions is presented for a solenoid magnet with orthotropic material properties. Analytic expressions are provided for the longitudinal, radial and axial stress and strain valid in a neighborhood of the mid-plane of a long solenoid. Calculations are made for an example magnet with multiple coil sections and external reinforcement. The results of generalized plane strain and finite element methods are compared with plane stress and plane strain analyses.**

I. Introduction and Summary

A three dimensional stress analysis of superconducting solenoid coils is formulated under assumptions which make it applicable to a region about the mid-plane of a long solenoid. The stress analysis of superconducting solenoids, as an important aspect of the overall design process, has been the subject of a number of publications [1]-[6]. Important among these have been the two dimensional plane stress and plane strain solutions [1], [2], [3]. Approximations of the three dimensional problem have been presented which allow calculation of the stress state beyond the mid-plane region. Solutions are obtained by neglecting shear throughout the coil, and not simply in the mid-plane region [4]. Equations are formulated including shear, and then solved by numerical techniques [5]. Detailed finite element solutions are readily achieved as the most accurate numerical method.

The present work provides an analytical formulation of the three dimensional state of stress and strain in a superconducting solenoid. A fundamental simplifying assumption in the analysis is that the shear stress in the local region of the coil under analysis is zero. In this respect, the analysis may be considered a direct extension to three dimensions of the two dimensional plane stress and plane strain solutions described previously, which also assume zero shear. The present formulation is valid in cases where significant shear occurs toward the end of a coil and takes into account the effect of that shear near the mid-plane.

A second fundamental assumption which leads to the present solution is that the axial strain in the mid-plane region of a long solenoid is constant through the coil along a radius, for fixed axial location. This assumption is equivalent to the statement that the strain associated with the bending deformation of the cylindrical shape of the coil is small in comparison with the other strains in the coil. The applicability of this assumption to the three dimensional stress analysis of superconducting solenoids has been recognized previously [6], [7]. The value of the axial strain, which is an output of the analysis, is the Generalized Plane Strain.

The present analysis addresses the averaged values of stress and strain computed on the assumption of average material properties in each separate region of coil windings and reinforcement. The relationship between the macromechanical, or averaged, material properties and the micromechanical, or constituent, material properties has been examined previously in detail [8]. The stress analysis is performed on the basis of the average properties. The stress in individual components of the winding composite, such as the conductor, may then be obtained from a knowledge of the local average stress. The full three dimensional analysis allows the complete orthotropic properties of the coil windings and, if present, reinforcement structure to be included.

The analytical formulation presented here treats individual coil sections of constant current density. Solutions are obtained for single coil sections, for single coil sections with an external reinforcement shell, and more generally for a set of radial sections with constant or zero current density in coil sections or reinforcement sections, respectively. Although reference is made to superconducting coils, the analysis applies generally to coils with sections having constant current density.

By way of example, the analysis is applied to a magnet which incorporates the features of coils with multiple sections and external reinforcement. The results of the generalized plane strain analysis is compared with the results of a finite element analysis, and also with results from the earlier plane stress and plane strain analyses. The comparisons provide an indication of the accuracy of the analytical techniques.

An extended report on this work includes the equations for the coefficients for various cases, additional examples, and formulation of the thermal problem [9].

II. Equations for Stress and Strain

The windings and reinforcement of a superconducting solenoid are treated as homogeneous, orthotropic, linear elastic materials. In the absence of shear, the relationship between stress and strain may be written in terms of the material parameters as

$$
\begin{aligned}
\varepsilon_\theta &= \frac{\sigma_\theta}{E_\theta} - \nu_{r\theta}\frac{\sigma_r}{E_r} - \nu_{z\theta}\frac{\sigma_z}{E_z} \\
\varepsilon_r &= -\nu_{\theta r}\frac{\sigma_\theta}{E_\theta} + \frac{\sigma_r}{E_r} - \nu_{zr}\frac{\sigma_z}{E_z} \\
\varepsilon_z &= -\nu_{\theta z}\frac{\sigma_\theta}{E_\theta} - \nu_{rz}\frac{\sigma_r}{E_r} + \frac{\sigma_z}{E_z}.
\end{aligned}
\tag{1}
$$

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Inverting these equations, the relationship between stress and strain may be written in terms of the stiffness matrix

$$\sigma_i = C_{ij}\varepsilon_j \quad i,j = 1,2,3. \tag{2}$$

The coordinates 1, 2, and 3 are in the directions θ, r, and z respectively. Following the traditional treatment, the radial displacement u is introduced as a primary variable. The displacement strain equations

$$\varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_r = \frac{du}{dr}, \tag{3}$$

together with the basic assumption that ε_z is constant, are introduced into (2) to give the components of stress as a function of the displacement.

The components of stress are related by the force balance equation

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + rJ_\theta B_z(r) = 0 \tag{4}$$

which, for zero shear, is independent of the axial stress. The assumption is made that the axial field is linear through the coil section as a function of the radius.

$$B_z(r) = B_c - C_0 r \tag{5}$$

Substitution of the stress components into (4) results in a second order differential equation for the displacement which has the general solution

$$u = D_1 r^k + D_2 r^{-k} + A_1\varepsilon_z r + A_2 r^2 + A_3 r^3 \tag{6}$$

where

$$\begin{aligned} A_1 &= -\frac{1}{(1-k^2)}\frac{(C_{23}-C_{13})}{C_{22}} \\ A_2 &= -\frac{1}{(4-k^2)}\frac{J_\theta B_c}{C_{22}} \\ A_3 &= \frac{1}{(9-k^2)}\frac{J_\theta C_0}{C_{22}} \end{aligned} \tag{7}$$

and

$$k^2 = \frac{C_{11}}{C_{22}}. \tag{8}$$

When k has the value 1, 2, or 3, the solution avoids the apparent singularity in the associated A_k by replacing the factor involving k^2 with the function $\ln r/2k$. In the following, non-singular values of k are assumed.

The expression for the displacement provides the strain through (3) and the stress through (2). The components of strain are

$$\varepsilon_\theta = D_1 r^{k-1} + D_2 r^{-k-1} + A_1\varepsilon_z + A_2 r + A_3 r^2 \tag{9}$$

$$\varepsilon_r = kD_1 r^{k-1} - kD_2 r^{-k-1} + A_1\varepsilon_z + 2A_2 r + 3A_3 r^2. \tag{10}$$

The resulting components of the stress are

$$\begin{aligned} \sigma_i = &(C_{i1}+kC_{i2})D_1 r^{k-1} + (C_{i1}-kC_{i2})D_2 r^{-k-1} \\ &+(C_{i1}+C_{i2})A_1\varepsilon_z + (C_{i1}+2C_{i2})A_2 r \\ &+(C_{i1}+3C_{i2})A_3 r^2 + C_{i3}\varepsilon_z. \end{aligned} \tag{11}$$

The determination of the unknown coefficients D is coupled to the determination of the strain ε_z. Applicable boundary conditions are zero radial stress at the inner and outer surfaces of a coil, and equal radial stress and displacement at the interface between sections within a coil. In addition, the axial force balance requires that the net axial stress taken over a section through the coil is equal to the applied axial load

$$\int_{a_1}^{a_{n+1}} 2\pi r\sigma_z dr = F_z. \tag{12}$$

These conditions yield a set of coupled linear equations which are solved for the coefficients and the axial strain.

III. Comparative Calculations

In order to examine the results obtained with a generalized plane strain analysis, a 15 T example magnet is used as the basis of comparative calculations. The 15 T magnet consists of two mechanically independent coils. Each coil contains two radial sections of different current density. In addition, the inner of the two coils has an external reinforcement section. The example magnet is shown in Fig. 1. The parameters of the example magnet are given in Table 1, where a_3 is the outer radius of the reinforcement region.

TABLE I
PARAMETERS FOR 15 T EXAMPLE MAGNET WITH TWO COILS, EACH WITH TWO RADIAL SECTIONS

Coil	a_1 (mm)	a_2 (mm)	a_3 (mm)	b (mm)	J (A/mm^2)
1a	100	126	---	250	102.36
1b	126	148	154	250	161.37
2a	157	170	---	250	155.55
2b	170	210	---	250	149.54

For reference, the material properties assumed for the analysis are given in Table 2. The two sections of the inner coil are assumed to contain Nb_3Sn conductor, and the outer coil sections NbTi conductor. The values listed in Table 2 are computed average properties for the coil winding composite [7].

Comparative calculations were made for the example magnet using the finite element program ABAQUS [10]. The distributed Lorentz force was accommodated within the finite element method by linking to a user supplied external subroutine [11]. This procedure allows for computation of fields and force densities during execution of the finite element program. Comparative calculations were also made using the earlier plane stress and plane strain analyses.

TABLE 2
MATERIALS PROPERTIES OF COIL WINDING COMPOSITE AND REINFORCEMENT

	Nb_3Sn Coils	NbTi Coils	Reinforcement
E_θ (GPa)	66.5	81.5	190.0
E_r (GPa)	41.4	45.4	190.0
E_z (GPa)	44.5	49.4	190.0
$G_{\theta r}$ (GPa)	14.7	16.0	73.0
$G_{\theta z}$ (GPa)	15.7	17.5	73.0
G_{rz} (GPa)	11.8	12.5	73.0
$\nu_{\theta r}$	0.341	0.341	0.305
$\nu_{\theta z}$	0.318	0.319	0.305
ν_{rz}	0.204	0.182	0.305

where $\nu_{ab} = -\varepsilon_b/\varepsilon_a$.

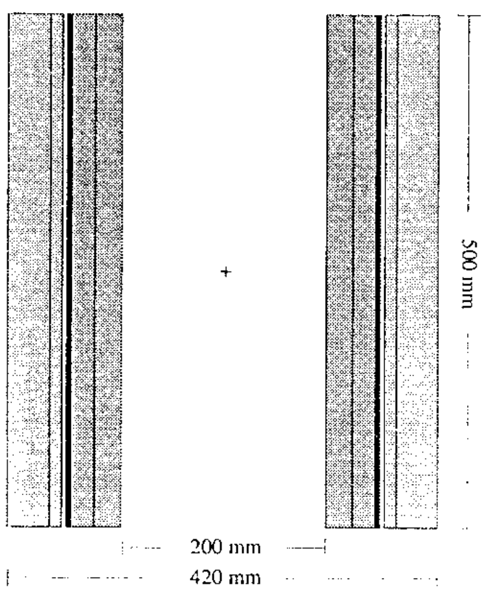

Fig. 1 15 Tesla example magnet with two coils, each with two radial sections.

The figures that follow are plots of stress and strain through the mid-plane of the example magnet. The two coil sections are distinctively shown and in some cases the results of the overbanding reinforcement clearly stand out as a sharp discontinuity.

The dominant stress component in solenoid magnets is in the tangential direction. As seen in Fig. 2, all of the computational methods considered here essentially gave the same tangential stress results.

From Fig. 3, a greater difference among the methods can be seen in the results for tangential strain. While the plane stress results agree more closely with the finite element than the plane strain results, there is essentially no difference between the generalized plane strain and finite element results. The discrepancy which does exist for the plane stress calculation should not be attributed to the zero shear assumption but rather to the non-physical treatment of the axial stress.

The agreement for the radial stress among the various methods is quite good, as Fig. 4 presents, but once again there are differences in the radial strain results as seen in Fig. 5. As with the tangential strain, there is little difference between the generalized plane strain and the finite element results.

While the dominant tangential stress is of interest for structural considerations, the transverse stress is also of great interest for strain sensitive conductors like Nb_3Sn. A primary objective of the generalized plane strain formulation is to provide an analytic method to compute transverse stress and strain including the axial components. The axial stress distribution for the example magnet is shown in Fig. 6. The generalized plane strain results agree well with the finite element results. By definition, the axial stress for the plane stress analysis is zero. In addition, the axial stress prediction for plane strain yields tensile rather than compressive results.

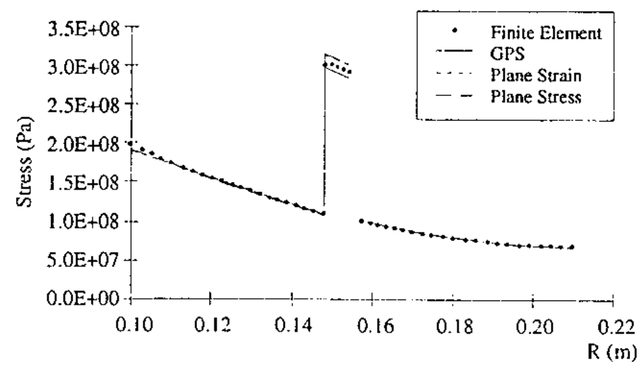

Fig. 2 Comparison of Tangential Stress Results.

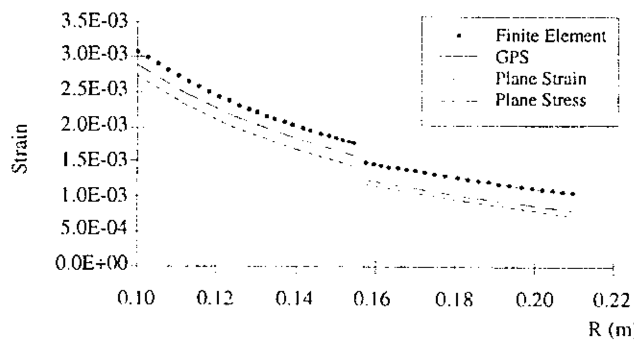

Fig. 3 Comparison of Tangential Strain Results.

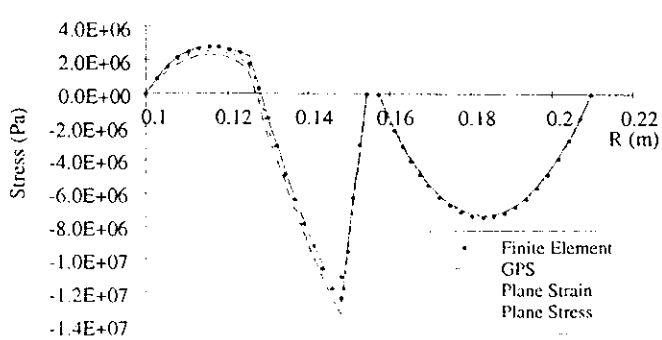

Fig. 4 Comparison of Radial Stress Results.

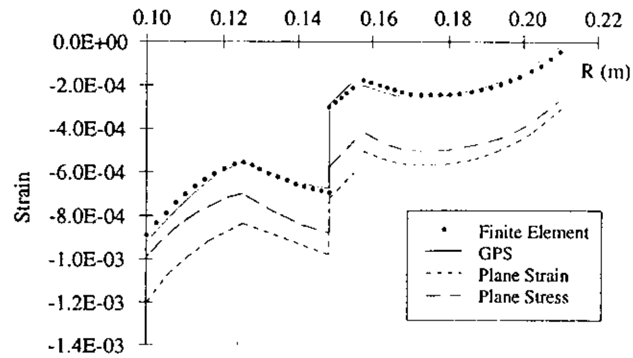

Fig. 5 Comparison of Radial Strain Results.

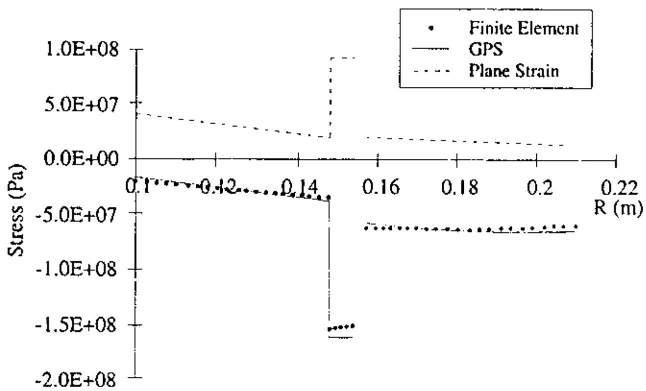

Fig. 6 Comparison of Axial Stress Results.

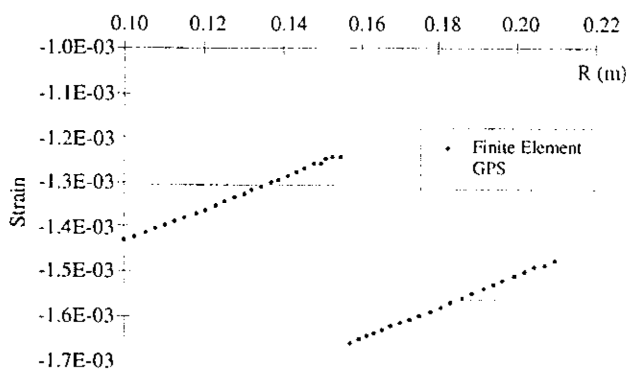

Fig. 7 Comparison of Axial Strain Results.

The degree to which the axial strain is approximated as a constant can be seen in Fig. 7. The actual strain may differ from constant with either positive or negative slope depending on the distortion of the coil as influenced by the configuration of nearby coils. Given the fact that a constant axial strain is the major simplifying assumption which leads to the other results, the agreement in the example is judged to be reasonably close.

IV Conclusion

The method of generalized plane strain, with the assumption of constant axial strain and the integral constraint on the axial stress, provides an accurate calculation of axial stress in long solenoid magnets. The method improves the accuracy of the tangential and radial strain over the previous plane stress and plane strain analyses.

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