IEEE TRANSACTIONS ON MAGNETICS, Vol. 30, No. 4, July 1994, pp. 2237-2240

Concepts of Plasticity in Solenoid Stress Analysis

H. Garmestani, M. R. Vaghar, and W. D. Markiewicz*
FAMU/FSU College of Engineering, Tallahassee, Florida 32306-2175
*National High Magnetic Field Laboratory, Florida State University, Tallahassee, Florida 32306-4005

***Abstract*—The concepts of plasticity are reviewed in the context of solenoid magnet design. The von Mises and Hill yield functions are introduced and related to flow rules. The derivation of equations for an elastic-plastic analysis of solenoids is discussed. Material properties are derived from Nb_3Sn conductors and used in example calculations. The results of elastic-plastic analyses are compared with those of elastic analyses under various assumptions.**

I. INTRODUCTION

The stress-strain curve for NbTi composite superconductor, although displaying some unique aspects, is typical of a linearly elastic, ductile material . This is not the case for superconductors which are formed by a high temperature heat treatment, such as the A-15 compounds and the ceramic superconductors. The stabilizer in these latter superconductors is annealed during the heat treatment. As a result, these composite superconductors exhibit non-linear stress-strain curves at relatively low values of applied stress. The stress analysis of a magnet depends on the mechanical properties of the conductor. For those conductors having a stress-strain curve which exhibits plastic characteristics, a stress analysis based on plasticity theory would appear to be most appropriate. Initial results of the application of plasticity theory to the stress analysis of a superconducting solenoid are presented here. The relevant characteristics of a non-linear stress-strain curve are reviewed. Typical stress-strain curves for Nb_3Sn conductors are surveyed. The formalism of plasticity theory is presented in the context of solenoid magnet design. Equations have been derived which give the stress and strain in solenoid coils for various yield criteria [1], [2]. The formulation of equations for the yield criteria of von Mises and Hill is summarized here. The equations are applied to example coils and the results of the elastic-plastic analysis are compared to the results of a linear elastic analysis.

II. STRESS-STRAIN CURVE

An idealization of a typical stress-strain curve obtained for a ductile material is shown in Fig. 1. The plot of applied stress versus total strain usually has an initial linear region in which the material is elastic and the strain is reversible as a function of applied stress. The slope of the initial linear region, from the origin to point a in the figure, is the Young's modulus E for the material. For emphasis and clarity, the modulus will be referred to here as the initial modulus. As the stress is increased past the point a, the stress-strain curve ceases to be linear, the material begins to yield, and the strain state is no longer reversible as a function of applied stress.

This work was supported by the National High Magnetic Field Laboratory through the NSF Cooperative Agreement #DMR-9016241 and the State of Florida. Manuscript received September 20. 1993.

Typical, ideal, behavior is shown in Fig. 1 with the return of the strain from point b to point c with removal of the applied load. The offset strain to point c, also refereed to as the non-recoverable strain, is the plastic component of strain. The strain between points c and d is the recoverable, or elastic, strain component. In practical applications, the yield stress for a material is usually defined in terms of that stress which corresponds to a given value of plastic strain, such as 0.2% off-set yield. In the terminology used in this paper, the yield point will refer to point a.

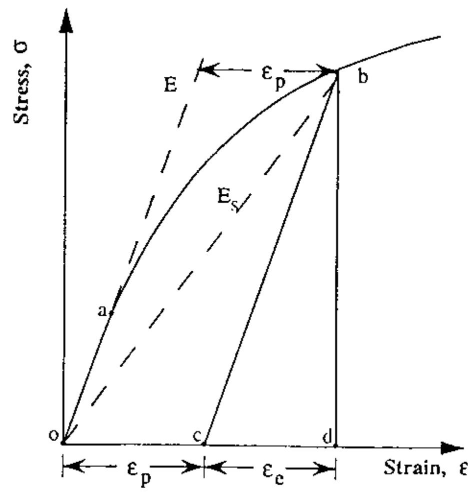

Fig. 1 Stress-strain curve for ductile material

At a point beyond yield, such as point b, the total strain is the sum of an elastic strain component and a plastic strain component, as shown in Fig. 1. In the stress-strain curve of an actual material, the behavior between points b and c exhibits some hysteresis. For applied stress beyond point b, the material will quickly resume the same stress-strain curve independent of the unloading cycle.

A further valuable concept is that of the secant modulus E_S, which is the slope of the line joining points o and b. The secant modulus is an effective modulus for an equivalent linearly elastic material which would exhibit the loading state b.

Stress-strain curves obtained at 4.2 K for a variety of Nb_3Sn superconductors are given in Fig. 2. The conductors are identified in Table 1. The extent to which a stress-strain curve is considered non-linear can depend on the portion of the curve which is of interest for a particular application. The critical current density of Nb_3Sn is strain sensitive, with a maximum current density typically in the neighborhood of 0.25% for longitudinal strain [3]. For a strain of this value, reference to Fig. 2 indicates that the total strain will include a significant fraction of plastic strain. This fact motivates the interest in examining the results of a stress theory which incorporates the plastic characteristics of the conductor.

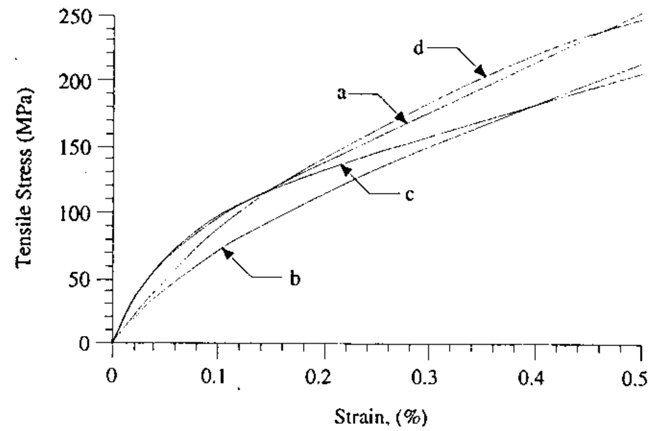

Fig. 2 Typical stress-strain curves for Nb_3Sn conductors.

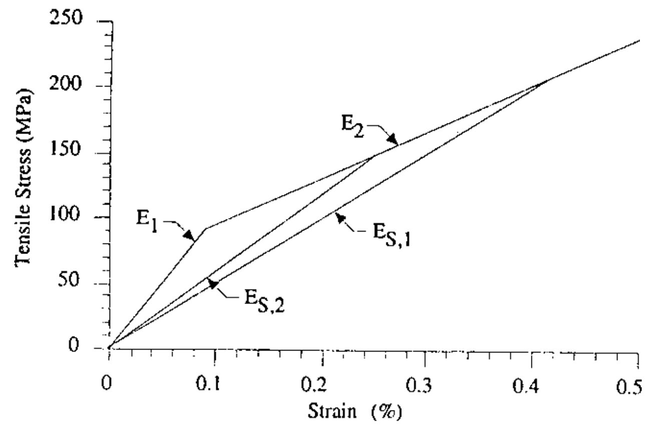

Fig. 3 Linearized conductor and effective stress-strain curve.

Although the strain dependence of the Nb_3Sn superconductor is well known, it is emphasized that the critical current density versus strain is measured during the initial loading of the conductor. The strain corresponds to the total strain measured from the origin in Fig. 1. The associated stress-strain curve is therefore the curve **oab** in Fig. 1, as opposed to curve **cb** for subsequent load cycles. It is noted that if the state of stress and strain along curve **oab**, for example at point *b*, were to be interpreted in a linear elastic model, the ratio of stress to strain is the secant modulus E_S and not the initial modulus.

TABLE I
Nb_3Sn CONDUCTORS

Conductor	Type	Reference
a	Bronze Process	4
b	MJR	5
c	Internal Tin	6
d	Tubular Tin	7

In order to simplify the analysis, a piecewise linear stress-strain curve, shown in Fig. 3, is used to approximate the characteristics of the conductor. The objective here is not to model a particular Nb_3Sn conductor, but rather to examine the implications of plasticity. The initial modulus E_1 and secondary modulus E_2 of the conductor, given in Table 2, are average values based on the curves of Fig. 2. The average properties of the windings, the tangential component of which is listed in Table 2, are computed from the conductor properties by the methods of composite materials. The secant moduli E_S shown in Fig. 3 are related to the examples.

III. Formulation of Equations

The theory of plasticity is formulated in terms of flow rules which give the increments of plastic strain in terms of applied stress. The elastic strain component of the total strain is related to the state of stress by the generalized Hooke's law. The state of elastic strain is determined by the state of stress. The state of plastic strain is, in general, determined by the history of the applied stress. It is therefore appropriate to have a formulation of plasticity in terms of the increments of plastic strain.

In the treatment of plasticity, it is useful to decompose the total stress tensor into a spherical stress tensor and a stress deviator tensor. The spherical stress component, which is associated with a uniform hydrostatic pressure, is generally considered to have no effect on yielding and plastic flow. It is the stress deviator tensor which is used as the fundamental quantity in plasticity.

The flow rule is a description of the yielding process in the material, and as such is dependent on the yield criterion or yield function for the material. The form of the yield function is reflected in the form of the flow rule. The flow rules associated with the yield functions of von Mises and Hill are introduced. The nature of the solution to the plasticity equations is described for different cases.

A. von Mises

For the well known von Mises yield criterion, the flow rule has the form given by the Prandtl-Reuss equation

$$d\varepsilon^{p}{}_{ij} = \frac{3}{2}\frac{d\varepsilon^{p}}{\sigma_e}S_{ij} \tag{1}$$

which relates the increments of plastic strain $d\varepsilon^{p}{}_{ij}$ to the components of the stress deviator tensor S_{ij}. The effective stress σ_e has the form of the von Mises yield function, which in the principle axis of the magnet coordinates is

$$\sigma_e{}^2 = \frac{1}{2}\left[(\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + (\sigma_r - \sigma_\theta)^2\right]. \tag{2}$$

When the effective stress exceeds the yield stress, the material begins to deform plastically. Above the yield stress, the components of stress continue to be related to the effective stress by Eq. (2). The effective stress provides the connection to the uniaxial tensile test, for which the effective stress reduces to the applied stress. A functional form similar to Eq. (2) exists for an effective plastic strain increment $d\varepsilon^p$ in terms of the increments of the components of plastic strain. The effective plastic strain reduces to the plastic strain in the uniaxial tensile test. The stress-strain curve therefore provides the relationship between the generalized concepts of effective stress and effective plastic strain.

For a plane strain condition, it is found that the components of the stress deviator tensor are proportional to the effective stress. The Prandtl-Reuss equations are integrated directly into an algebraic equation in the strains only. It then is possible to derive a linear differential equation for the tangential component of plastic strain

$$C_1 \frac{d\varepsilon_\theta{}^p}{dr} + 2C_1 \frac{\varepsilon_\theta{}^p}{r} + C_2 \frac{1}{r}\int \frac{\varepsilon_\theta{}^p}{r} dr + C_3 \frac{1}{r} + C_4 \frac{\ln r}{r} + C_5 + C_6 r = 0. \tag{4}$$

The coefficients C depend on the material properties, the current density, the windings, and the magnetic field strength. The equation may be solved in closed form, and the expressions for the stress in the plastic region of the stress-strain curve follow directly.

For a plane stress assumption with the von Mises yield function, the derivation leads to non-linear differential equations. The numerical solution which is obtained is a special case of the more general Hill yield function.

B. Hill

The more general formulation of plasticity is based of a flow rule of the form

$$\mathrm{d}\varepsilon_{ij}{}^p = \mathrm{d}\lambda \frac{\partial f(\sigma_{ij})}{\partial \sigma_{ij}} \tag{5}$$

where the increments of plastic strain are related through the scalar differential to $d\lambda$ to the derivatives of the yield function. A specific form of the flow rule is obtained by substituting a given yield function into (5).

For an orthotropic material, the von Mises yield function is generalized to Hill's function which, in the principle magnet coordinates is written

$$2f(\sigma_{ij}) = F(\sigma_\theta - \sigma_r)^2 + G(\sigma_r - \sigma_z)^2 + H(\sigma_z - \sigma_\theta)^2 \tag{6}$$

where F, G and H are related to the tensile yield stresses in the principal directions. An epoxy impregnated magnet winding is nearly transversely isotropic, and this assumption is made in the following. For a transverse isotropic material, the Hill function reduces to

$$\sigma_e{}^2 = \frac{1}{2}\left[(\sigma_\theta - \sigma_r)^2 + \eta(\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\theta)^2\right] \tag{7}$$

where at yield, the effective stress σ_e is equal to the tangential yield stress. The parameter η is a measure of the anisotropy and may be written in terms of the yield stress as

$$\eta = \frac{G}{F} = 2\left(\frac{\sigma_{y\theta}}{\sigma_{yr}}\right)^2 - 1. \tag{8}$$

A magnet is an example of proportional loading, in which all the components of stress, and therefore the effective stress, increase at a rate proportional to their final values. In the case of proportional loading, the differential form of the flow rule may be integrated directly to give an algebraic equation for the plastic strain components. Here, as opposed to the von Mises case for plane strain, these equations contain the components of the stress. It is possible, by using Hooke's law, the force balance equation, the constitutive equation, and the Hill yield function, to eliminate the strains to arrive at two coupled non-linear first order differential equations for the components of the stress.

$$\begin{aligned} \frac{d\sigma_\theta}{dr} &= F_1(\sigma_\theta, \sigma_r) \\ \frac{d\sigma_r}{dr} &= F_2(\sigma_\theta, \sigma_r) \end{aligned} \tag{9}$$

These differential equations are solved using numerical methods.

IV. Example Calculations and Results

The elastic-plastic analysis is applied to two example coils. The parameters of the coils are shown in Table 2 where B_1 and B_2 are the axial field at a_1 and a_2. While the coils occupy the same radial depth and produce the same field, the second coil has as external reinforcement section and a proportionally higher current density. The tangential components of the elastic moduli used in the calculations are given in Table 3 where the values for the conductor are obtained from Fig. 3

Calculations are first made assuming isotropic yield conditions and using the von Mises and also the Tresca yield functions. These calculations are compared with the results of elasticity theory using both the initial modulus and a selected secant modulus. The implications of anisotropic yield are then examined with the Hill function.

TABLE 2
Parameters for Example Coil

Coil	a_1 (mm)	a_2 (mm)	a_3 (mm)	B_1 (Tesla)	B_1 (Tesla)	J (A/mm^2)
1	130	158	—	12.5	8.0	125
2	130	152	158	12.5	8.0	159

TABLE 3
Tangential Components of the Elastic Modulus for Conductor and Coil

	$E_{\theta,1}$ (GPa)	$E_{\theta,2}$ (GPa)	$E_{\theta,s1}$ (GPa)	$E_{\theta,s2}$ (GPa)
Conductor	100.0	36.0	58.8	49.5
Coil	81.3	33.3	50.4	43.4

For the first example coil, without reinforcement, the tangential stress is seen in Fig. 4 to be independent of the method of computation, and independent of the modulus. The tangential strains for the plasticity models agree well with one another in Fig. 5. The tangential strain predicted by an elastic calculation using the initial modulus is significantly and deceptively lower. When the computed stress, however, is used to select the secant modulus, E_{S1}, from the stress-strain curve in Fig. 3, the elastic calculation using this secant modulus is seen in Fig. 5 to agree well with the elastic-plastic results.

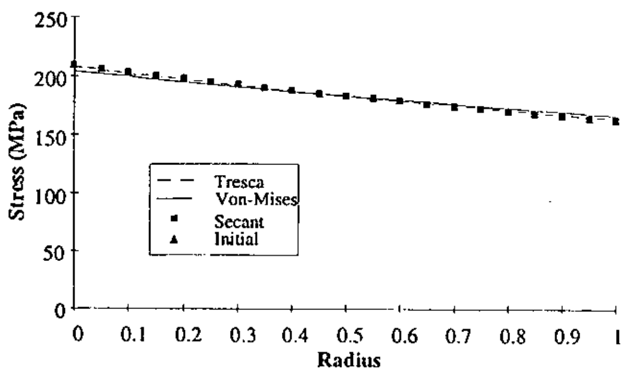

Fig. 4 Tangential Stress without Reinforcement

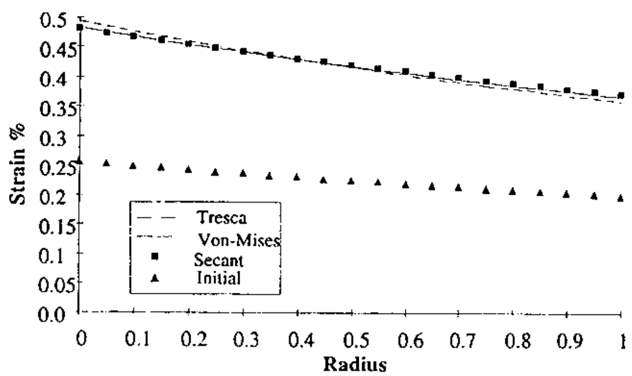

Fig. 5 Tangential Strain without Reinforcement

A similar situation exists for a coil with external reinforcement. As seen in Fig. 6, the elastic-plastic results agree for the Tresca and von Mises yield functions, and an elastic calculation based on the initial modulus gives lower values of strain. The average value of strain for the elastic-plastic result allows selection of a secant modulus E_{S2}, from Fig. 3. The elastic calculation based on this secant modulus agrees well with the elastic-plastic result.

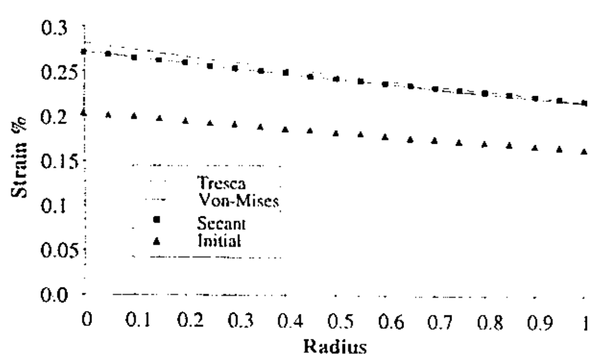

Fig. 6 Tangential Strain with Reinforcement

The implications of an anisotropic yield stress are shown in Fig. 7, where the Hill yield function is used with a range of values for the ratio of yield stress in the tangential to the radial direction. For the same value of tangential yield stress, the values of tangential strain depend on the yield stress ratio as shown in the figure.

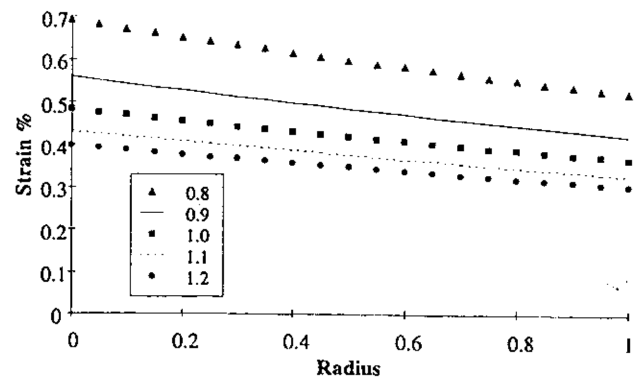

Fig. 7 Tangential strain distribution for various tangential to radial yield stress ratios $\sigma_{y\theta}/\sigma_{yr}$.

V. Conclusion

It has been shown that elastic-plastic equations for the analysis of stress in solenoid magnets can be derived for a number of yield functions including, most importantly, Hill's function. The initial application of these equations has emphasized the simple but important fact that the stress-strain curve of Nb_3Sn is sufficiently non-linear that an elastic calculation based on the initial modulus will be significantly in error. It is interesting to see the agreement between elastic calculations based on a self-consistent secant modulus and the elastic-plastic calculations, at least for the tangential values reported here. The potential for elastic-plastic theory to contain new and important information for magnet stress analysis is indicated by the results obtained for anisotropic yield stress. The winding composite is a complex, intrinsically elastic-plastic composite which simultaneously has components in the plastic and elastic state. Knowledge of the elastic-plastic material properties of the composite is required as input for further analysis.

References

[1] H. Garmestani, W.D. Markiewicz, and M.R. Vaghar, "Elastic-Plastic Analysis of Superconducting Solenoids," Internal Report #65, Tallahassee, FL: National High Magnetic Field Laboratory, December 1992.

[2] H. Garmestani, M.R. Vaghar, W.D. Markiewicz, and N. Chandra, "Elastic-Plastic Analysis of Anisotropic Work Hardening Cylinder with Body Force," To be published.

[3] J.W. Ekin, "Strain scaling law for flux pinning in practical superconductors, Part 1: basic relationships and application to Nb_3Sn conductors," Cryogenics, vol. 20, pp. 611 (1980)

[4] W. Specking, M. Klemm, R. Flükiger, P. Bruzzone, and M. Ricci, "The effect of static and cyclic axial strain on I_c of cable in conduit net subcables", *IEEE Trans. on Magnetics*, vol. 27, no. 2, pp. 1825-1828, March 1991.

[5] W. Specking, A. Nyilas, A. Ulbricht, P. Komarek, and R. Flükiger, "Performance of a 'react and wind' 12 T KfK-NET-TF subsize conductor under static and cyclic axial strain", *IEEE Trans. on Magnetics*, vol. 27, no. 2, pp. 1912-1915, March 1991.

[6] L.T. Summers, private communication.

[7] E. Gregory, L.R. Motowidlo, G.M. Ozeryansky, and L.T. Summers, "High Strength Nb_3Sn Conductors for High Magnetic Field Applications," *IEEE Trans. on Magnetics*, vol. 27, no. 2, pp. 2033-2036, March 1991.